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Higher-derivative terms in $N = 2$ supersymmetric effective actions

Philip C. Argyres, Adel M. Awad, Gregory A. Braun and F. Paul Esposito

Department of Physics, University of Cincinnati
Cincinnati OH 45221-0011, USA
E-mail: argyres@physics.uc.edu, adel@physics.uc.edu, braun@physics.uc.edu, esposito@physics.uc.edu

Abstract: We show how to systematically construct higher-derivative terms in effective actions in harmonic superspace despite the infinite redundancy in their description due to the infinite number of auxiliary fields. Making an assumption about the absence of certain superspace Chern-Simons-like terms involving vector multiplets, we write all 3- and 4-derivative terms on Higgs, Coulomb, and mixed branches. Among these terms are several with only holomorphic dependence on fields, and at least one satisfies a non-renormalization theorem. These holomorphic terms include a novel 3-derivative term on mixed branches given as an integral over 3/4 of superspace. As an illustration of our method, we search for Wess-Zumino terms in the low energy effective action of $N = 2$ supersymmetric QCD. We show that such terms occur only on mixed branches. We also present an argument showing that the combination of space-time locality with supersymmetry implies locality in the anticommuting superspace coordinates of for unconstrained superfields.

Keywords: Extended Supersymmetry, Supersymmetric Effective Theories.

Contents

1. Introduction and auxiliary fields in derivative expansions

Certain higher-derivative terms in the effective actions of four dimensional gauge theories with extended supersymmetry have been shown to be not renormalized [1, 2]. Another class of non-renormalized higher derivative terms are the Wess-Zumino (WZ) terms, shown to exist, though their fully supersymmetric form was not determined, by a one-loop calculation in [3] and by an anomaly matching argument [4] for $N = 4$ supersymmetric effective actions. They must therefore also exist in $N = 2$ effective actions. In this paper we will derive the fully supersymmetric form of these $N = 2$ supersymmetric WZ terms, (they were already found for $N = 1$ supersymmetric effective actions in [5]). In doing so, we will develop the tools to perform a systematic exploration of higher-derivative terms in $N = 2$ effective actions, and will carry out this exploration to construct all terms up to and including 4 derivatives, many of which are constrained by new non-renormalization theorems.

Although higher-derivative terms in the low energy effective actions of four dimensional gauge theories with extended supersymmetry have received some attention [6]–[22], no systematic exploration of the $N = 2$ case has been done. This may be due to the difficulty of generating higher-derivative terms with extended supersymmetry. In an on-shell and/or component formalism, the problem is that one must self-consistently correct the supersymmetry transformation rules order by order in the derivative expansion at the same time that one tries to construct the supersymmetry-invariant higher-order term in the action. The solution to this problem is to use an off-shell superfield formulation so that the supersymmetry transformations are independent of the form of the action. In this case, it only remains to list all the supersymmetry invariants with a given number of derivatives. A prescription for generating all possible such terms might only exist if the superfields are unconstrained; the constrained case is unclear. Harmonic superspace [23] gives such an unconstrained superfield formulation for $N = 2$ supersymmetry. We therefore use the harmonic superspace formalism in this paper. An important feature of harmonic superspace is that, in addition to the usual space-time directions described by coordinates x^μ and Grassmann-odd directions with spinor coordinates θ^i_α, there is also a 2-sphere described by commuting harmonic coordinates $u^\pm$; see, e.g., [24] or section 3 below for a review of harmonic superspace.

The low energy effective action at a generic vacuum of $N = 2$ gauge theory includes only massless U(1) vector multiplets and massless neutral hypermultiplets, since charged hypermultiplets generically get masses by the Higgs mechanism. We call the set of those vacua with only massless neutral hypermultiplets the "Higgs branches", those with only U(1) vector multiplets the "Coulomb branch", and those vacua with both kinds of multiplets the "mixed branches" [25, 26, 27]. Thus the low energy propagating fields are massless neutral scalars ϕ and spinors ψ_α, and U(1) vectors A^μ.

We organize the terms in a low energy effective action by a kind of scaling dimension which essentially counts the number of derivatives, and we refer to this counting as the "dimension" and denote it by square brackets. It should be noted that it is not the same as the scaling dimension used in the renormalization group analysis of fluctuations around a given vacuum. The leading term[1] in the effective lagrangian for the scalar fields is, schematically,

$$g(\phi)\partial^\mu\phi\partial_\mu\phi\,. \tag{1.1}$$

Since this is a 2-derivative term we assign it dimension 2, so we must assign $[\partial_\mu] = 1$ and $[\phi] = 0$. Supersymmetry then determines the dimensions of the other fields: the supersymmetry algebra implies that $-[\theta] = [d\theta] = [\partial/\partial\theta] = 1/2$, the normalization of the harmonic sphere $u^\pm$ coordinates (e.g., eq. 3.2 below) implies that $[u] = [\partial/\partial u] = [du] = 0$, and examination of the component expansion of hypermultiplet and vector multiplet superfields then implies that $[\psi_\alpha] = 1/2$ and $[A_\mu] = 0$.

A drawback of the harmonic superspace formalism, or any unconstrained off-shell superfield formalism describing hypermultiplets, is that its superfields include an infinite number of auxiliary component fields. This implies that at each order of the derivative expansion of an effective action, we are able to write an infinite number of terms of that order in harmonic superspace, even though, when written in terms of propagating component fields (i.e., after substituting for the auxiliary fields by their equations of motion), there is only a finite number of such terms.[2] This infinite redundancy in the harmonic superfield

[1] We are considering here theories without Fayet-Iliopoulos terms, e.g. any $N = 2$ gauge theory whose microscopic description involves only semi-simple gauge groups.

[2] Note that we do not count the infinite number of coefficient functions of the dimensionless propagating scalars, such as $g(\phi)$ in (1.1) as separate terms.

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formalism appears because the lagrangian can be non-local with respect to the harmonic 2-sphere coordinates, or, equivalently, because terms with arbitrarily many $\partial/\partial u$ derivatives enter at each order of the derivative expansion. There is no physical requirement of locality in the auxiliary u variables.

(The same argument could be made for the auxiliary Grassmann variables θ in superspace: Since there is no physical requirement that effective actions be local in the Grassmann coordinates, doesn't this mean that every superspace description, not just harmonic superspace, suffers from this kind of redundancy? It turns out that the combination of locality in space-time together with supersymmetry invariance and the anticommuting nature of Grassmann coordinates implies that any term in the effective action of unconstrained superfields can be written as the integral of a local functional in the Grassmann coordinates. This argument is presented in appendix A below, and will be useful in the discussion of $N = 2$ vector multiplets in section 5.)

A worry is then that this redundancy following from non-locality in the $u^{\pm}$ coordinates makes the harmonic superspace formalism useless for systematic derivative expansions of effective actions, for there is no simple way of listing or parameterizing terms with arbitrary $\partial/\partial u$ derivative dependence. General considerations involving the nature of auxiliary fields and of derivative expansions, however, imply that there is a finite, order-by-order, procedure for constructing all terms in the effective action given the leading term, as we will now show.

Consider a theory with some propagating fields and some auxiliary fields which we collectively denote by p and a, respectively. Suppose we are given some leading (2-derivative) action $S_2(p,a)$, such that the $\partial_a S_2 = 0$ equations of motion determine the auxiliary fields in terms of the propagating fields and their derivatives,

$$a = a_2(\partial, p)\,, \tag{1.2}$$

as is the case with the 2-derivative hypermultiplet and vector multiplet actions in harmonic superspace. Thus, a_2 are the functions satisfying

$$\partial_a S_2|_{a=a_2} = 0\,. \tag{1.3}$$

Now the general effective action is a sum of contributions S_n with $n \geq 2$ derivatives:

$$S = S_2 + \ell S_3 + \ell^2 S_4 + \cdots\,, \tag{1.4}$$

where ℓ is the cut-off length scale which organizes the derivative expansion. In an effective action expansion, we develop the fields in a power series expansion in ℓ. In particular, the solution to the equations of motion for the auxiliary fields,

$$0 = \partial_a S = \partial_a S_2 + \ell \partial_a S_3 + \ell^2 \partial_a S_4 + \cdots\,, \tag{1.5}$$

following from (1.4), will be the leading piece (1.2) plus corrections of order ℓ:

$$a = a_2(\partial,p) + \ell a_3(\partial,p) + \ell^2 a_4(\partial,p) + \cdots\,. \tag{1.6}$$

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Plugging (1.6) back into (1.5) and expanding in powers of ℓ, we then determine a_n in terms of the a_m, $m < n$:

$$\begin{aligned} a_3 &= -\left(S_2''|_2\right)^{-1} S_3'|_2, \\ a_4 &= -\left(S_2''|_2\right)^{-1}\left(S_4'|_2 + a_3 S_3''|_2 + \frac{1}{2}a_3^2 S_2'''|_2\right), \\ & \quad \cdots, \end{aligned} \tag{1.7}$$

where primes denote derivatives with respect to a and $|_2$ means evaluate at $a = a_2$. Substituting (1.6) back into the full action (to eliminate the auxiliary fields) and expanding in powers of ℓ^2, we have

$$\begin{aligned} S &= S_2|_2 + \ell\left[a_3 S_2'|_2 + S_3|_2\right] + \ell^2\left[a_4 S_2'|_2 + \frac{1}{2}a_3^2 S_2''|_2 + a_3 S_3'|_2 + S_4|_2\right] + \cdots \\ &= S_2|_2 + \ell S_3|_2 + \ell^2\left[S_4 - \frac{1}{2}S_3'(S_2'')^{-1}S_3'\right]|_2 + \cdots, \end{aligned} \tag{1.8}$$

where in the second line we have used (1.3) and (1.7). The crucial point is that when the auxiliary fields are eliminated by substitution, the n-derivative piece of the action, appearing at order ℓ^{n-2}, depends on S_n only through $S_n|_2$, i.e., with the auxiliary fields evaluated at their values $a = a_2$ determined from the leading (S_2) term in the derivative expansion. This means that in classifying the S_n terms, *it is sufficient to substitute in the auxiliary component fields in terms of the propagating components using (1.2).* Since there are only a finite number of terms in S_n when written in terms of the propagating components, it follows that only a finite number of superfield expressions need be examined at this order, despite their infinite number of possible forms. In particular, this means that there will be identities relating the superfields evaluated at $a = a_2$ to some number of their $\partial/\partial u$ derivatives which can be used to truncate the $\partial/\partial u$ expansion at each order in the space-time derivative expansion. Explicit examples of this procedure appear in sections 4 and 6 below.

The above argument is not enough to assure us that we can use the harmonic superspace techniques to perform systematic derivative expansions. Two further problems may arise. First, the above argument assumed that the leading, 2-derivative, term was already given, and did not tell us how to remove the infinite redundancy in its harmonic superspace description. Fortunately this hard work has already been done for us (see, e.g., Chapters 7 and 11 of [24]): superspace expressions for all $N = 2$ supersymmetric 2-derivative terms for U(1) vector multiplets and neutral hypermultiplets have been found.

A second potential problem is that a systematic derivative expansion can only be carried out if the superfields all have non-negative dimension. For suppose that a field had negative dimension: then a term of given overall dimension may have an arbitrarily large number of positive-dimension derivatives as well as negative-dimension fields. This problem does not arise for the hypermultiplet superfield q^+ since it turns out to have dimension 0. The gauge invariant field strength superfield W for the vector multiplets also has dimension 0. However, it is a constrained superfield as it must satisfy the Bianchi identities. The unconstrained superfield is the vector potential superfield V^{++} which has dimension -1. Actions built from the V^{++} must be gauge invariant. This acts as a powerful restriction on the ways in which the V^{++} can enter. In particular, the question arises whether there can exist Chern-Simons-like terms in $N = 2$ harmonic superspace, i.e., do there exist gauge invariant U(1) vector multiplet terms which cannot be written solely in terms of the field strength multiplets W? If not, then we can just work with the dimension 0 field strength superfield W. The existence of superspace Chern-Simons-like terms is a difficult algebraic question; they are known to occur, for example, in $N = 3$ harmonic superspace [28, 29, 30, 24]. In section 5 below we show that in $N = 2$ harmonic superspace any gauge-invariant term written in terms of the potential superfield V^{++} can be rewritten solely in terms of derivatives of the field strength superfield W, at the expense of introducing non-localities in the Grassmann coordinates. The argument of appendix A, showing locality in the Grassmann directions of superspace, fails when the superfields obey extra constraints, in this case the Bianchi identities. *Thus we show that the existence of superspace Chern-Simons-like terms is equivalent to the existence of supersymmetric expressions involving field-strength superfields non-local in the Grassmann coordinates.* Due to the nilpotent nature of Grassmann variables, the number of such possible non-local terms is finite at each order in the derivative expansion. We will explore the possibility of superspace Chern-Simons-like terms with three or four derivatives elsewhere [31], and, for the purposes of this paper, we will assume they do not occur.

2. Summary of results

Modulo the question of the existence of superspace Chern-Simons-like terms, we find the following possible harmonic superspace forms for 3- and 4-derivative terms in the low energy effective action of $N = 2$ supersymmetric theories. On the Higgs branches, with only neutral hypermultiplets q_I^+ and their complex conjugates $q_{\overline{I}}^-$, there are no 3-derivative terms, and two types of 4-derivative terms (harmonic superspace notation is reviewed in section 3 below):

$$\begin{aligned} S_{4a}^H &= \int du\, d^4x\, d^2\theta^+\, d^2\overline{\theta}^+\, \partial^\mu q_I^+ \partial_\mu q_J^+\, B^{IJ}(q_K^+; u^\pm, D^{++}) + \text{c.c.}\,, \\ S_{4b}^H &= \int du\, d^4x\, d^2\theta^+\, d^2\overline{\theta}^+\, d^2\theta^-\, d^2\overline{\theta}^-\, \Gamma(q_I^+, q_{\overline{I}}^-; u^\pm, D^{\pm\pm})\,. \end{aligned} \tag{2.1}$$

Similarly, on the Coulomb branch, with only U(1) field strength vector multiplets W_a and their complex conjugates $\overline{W}_{\overline{a}}$, we find

$$\begin{aligned} S_{4a}^C &= \int d^4x\ d^2\theta^+\, d^2\theta^-\ \partial^\mu W_a \partial_\mu W_b\ \mathcal{G}^{ab}(W_c) + \text{c.c.}\,, \\ S_{4b}^C &= \int d^4x\ d^2\theta^+\, d^2\overline{\theta}^+\, d^2\theta^-\, d^2\overline{\theta}^-\ \mathcal{H}(W_a, \overline{W}_{\overline{a}})\,. \end{aligned} \tag{2.2}$$

On the mixed branches there are both 3-derivative and 4-derivative terms:

$$S_3^M = \int du\, d^4x\ d^2\theta^+\, d^2\overline{\theta}^+\, d^2\theta^-\ F(q_I^+, W_a; u^\pm, D^{++}) + \text{c.c.}\,,$$

$$
\begin{aligned}
S^M_{4a} &= \int du\, d^4x\, d^2\theta^+\, d^2\overline{\theta}^+\, d^2\theta^-\, D^+W_a \cdot D^+W_b\; G^{ab}(q^+_I, W_c; u^\pm, D^{++}) + \text{c.c.}\,,\\
S^M_{4b} &= \int du\, d^4x\, d^2\theta^+\, d^2\overline{\theta}^+\, d^2\theta^-\, \overline{D}^-(D^{++})^n q^+_I \cdot \overline{D}^-(D^{++})^m q^+_J \times\\
&\quad \times G^{IJ}_{nm}(q^+_K, W_a; u^\pm, D^{++}) + \text{c.c.}\,,\\
S^M_{4c} &= \int du\, d^4x\; d^2\theta^+\, d^2\overline{\theta}^+\, d^2\theta^-\, d^2\overline{\theta}^-\;\; H(q^+_I, q^-_{\overline{I}}, W_a, \overline{W}_{\overline{a}}; u^\pm, D^{\pm\pm})\,.
\end{aligned} \tag{2.3}
$$

For each of the mixed branch terms given as integrals over 3/4 of superspace (S^M_3, S^M_{4a}, and S^M_{4b}), there is another term given by an integral over a different three-quarters of superspace, for example:

$$
S'^M_3 = \int du\, d^4x\, d^2\theta^+\, d^2\overline{\theta}^+\, d^2\overline{\theta}^-\;\; F'(q^+_I, \overline{W}_{\overline{a}}; u^\pm, D^{++}) + \text{c.c.} \tag{2.4}
$$

The terms S^H_{4a}, S^C_{4a}, S^M_3, and $S^M_{4a,b}$ do not seem to have been noted elsewhere in the literature. They depend only on the analytic hypermultiplets and chiral vector multiplets and not their complex conjugates. The holomorphic nature of these terms suggests that they might be determined non-perturbatively using arguments similar to those of [25, 26].

Indeed, in $N = 2$ superQCD, where the strong coupling scale Λ can be thought of as the lowest component of a field strength vector superfield W, S^H_{4a} can get no Λ-dependent quantum corrections since it cannot involve any W's. Thus S^H_{4a} satisfies a non-renormalization theorem. (On the other hand, S^H_{4b} *can* get quantum corrections because we can add W-dependence to it as in S^M_{4c}. In this sense, for the puposes of deriving non-renomalization theorems, we should think of S^H_{4b} and S^C_{4b} as special cases of S^M_{4c}.)

Similarly, the Coulomb branch holomorphic 4-derivative term S^C_{4a} can only get quantum corrections holomorphic in Λ, i.e., only one loop and instanton corrections. Note that when there is only a single vector multiplet, S^C_{4a} can be rewritten using the Bianchi identity as an S^C_{4b} term; see the discussion after (5.13) below. Thus examples of S^H_{4a} terms only occur with two or more vector multiplets.

Finally, the holomorphic 3-derivative terms on the mixed branch, S^M_3 and S'^M_3, are of special interest since they also only get one loop and instanton corrections, and they give the entire leading correction to the mixed branch low-energy physics. They describe a derivative coupling between the hypermultplet scalars and the vectormultiplet photons; see (6.5) below.

The expressions in (2.1) and (2.3) are non-local in the $u^\pm$ variables, since they involve infinitely many $D^{\pm\pm}$ derivatives in gerneral. However, we will show that only a certain finite number of combinations of those derivatives may act on any given $q^\pm$ field in these expressions. For example, we show below that if the leading 2-derivative term for the hypermultiplets describes free hypermultiplets, then only the combinations q^+, $D^{++}q^+$, and $(D^{++})^2q^+$ may appear in S^H_{4a}, S^M_3, S^M_{4a}, and S^M_{4b}; while the non-holomorphic terms S^H_{4b} and S^M_{4c} can be taken to depend only on the combinations $(D^{--})^3q^+$, $(D^{--})^2q^+$, $D^{--}q^+$, q^+, $D^{++}q^+$, and $(D^{++})^2q^+$, and their complex conjugates involving q^-. Thus, in terms of these sets of fields, the effective actions are local in the harmonic superspace variables $u^\pm$ as well as the x^μ. For a more general 2-derivative term, the description of this finite set of hypermultiplet fields is more complicated, though it can be derived in principle. Furthermore, using this characterization of 4-derivative terms we show that WZ terms can only occur on mixed branches in $N = 2$ effective actions.

The outline of the rest of this paper is as follows. In section 3 we briefly review the harmonic superspace formalism. In section 4 we characterize the 4-derivative terms made out of hypermultiplets and we show that no WZ terms can be constructed purely from hypermultiplets. In section 5 we do the same for U(1) field strength vector multiplets, where we also discuss the problem of superspace Chern-Simons-like terms. Finally, in section 6 we characterize the 3- and 4-derivative terms with both vector and hypermultiplets, and construct WZ terms.

3. Harmonic superspace and notation

We briefly summarize harmonic superspace formalism following the notation and conventions of [24]. $N = 2$ supersymmetry without central charges has two fermionic generators, Q^i, $i = 1, 2$, satisfying $\{Q^i_\alpha, \overline{Q}_{\dot\alpha j}\} = 2\delta^i_j \sigma^\mu_{\alpha\dot\alpha} P_\mu$, with the other anticommutators vanishing. Since we need only consider neutral fields in the low energy effective action, there are no central charges. Harmonic superspace allows an unconstrained superfield formulation of $N = 2$ supersymmetry by permitting an infinite number of auxiliary fields. This is done in a superspace consisting of a standard superspace with coordinates $\{x^\mu, \theta^i_\alpha, \overline{\theta}_{\dot\alpha i}\}$ extended by two even coordinates on an additional 2-sphere. The θ's are Grassmann spinor coordinates satisfying the complex conjugation rule

$$\overline{\theta^i_\alpha} = -\overline{\theta}_{\dot\alpha i} \,. \tag{3.1}$$

The additional 2-sphere is conveniently coordinatized by introducing harmonic SU(2) group coordinates $u^\pm_i$. Here i is an SU(2) index, while the $\pm$ indices refer to the diagonal $\mathrm{U}(1) \subset \mathrm{SU}(2)$ charge. The $u^\pm$ variables satisfy the following basic identity,

$$u^+_i u^-_j - u^+_j u^-_i = \epsilon_{ij} \,, \tag{3.2}$$

along with the complex conjugation rule

$$\overline{u^\pm_i} = \mp u^{i\mp} \,. \tag{3.3}$$

Restriction to the coset sphere $S^2 = \mathrm{SU}(2)/\,\mathrm{U}(1)$ is realized by having all physical expressions be U(1) neutral. (Note that the SU(2) indices $i, j, k, \ldots$ are raised and lowered with the anti-symmetric ϵ^{ij} tensor defined by $\epsilon_{12} = -\epsilon^{12} = 1$, so that $a^i = \epsilon^{ij} a_j$, $a_i = \epsilon_{ij} a^j$.) Thus the full harmonic superspace consists of the space-time, Grassmann, and harmonic coordinates $\{x^\mu, \theta^i_\alpha, \overline{\theta}_{\dot\alpha i}, u^\pm_i\}$.

A basic assumption of the harmonic superspace formalism is that all fields are harmonic functions on the sphere, which is to say they are given by a power series expansion in the $u^\pm_i$ coordinates. Due to the identity (3.2), any product of $u^\pm_i$'s can be rewritten as a sum of terms each completely symmetric on SU(2) indices. For example, the expansion for a field of U(1) charge $+1$ will have the unique expansion $f^+ = f^i u^+_i + f^{(ijk)} u^+_i u^+_j u^-_k + \cdots$.

The usual superspace covariant derivatives are introduced

$$D^i_\alpha \equiv \frac{\partial}{\partial\theta^\alpha_i} + i\overline{\theta}^{\dot\alpha i}\,\partial\!\!\!/_{\alpha\dot\alpha}\,, \qquad \overline{D}_{\dot\alpha i} \equiv -\frac{\partial}{\partial\overline{\theta}^{\dot\alpha i}} - i\theta^\alpha_i\,\partial\!\!\!/_{\alpha\dot\alpha}\,, \tag{3.4}$$

where $\partial\!\!\!/_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha}\partial_\mu$, satisfying the $N=2$ algebra $\{D^i_\alpha, \overline{D}_{\dot\alpha j}\} = -2i\delta^i_j\,\partial\!\!\!/_{\alpha\dot\alpha}$. On the sphere we introduce derivatives

$$D^{++} \equiv u^+_i\frac{\partial}{\partial u^-_i}\,, \qquad D^{--} \equiv u^-_i\frac{\partial}{\partial u^+_i}\,, \qquad D^0 \equiv u^+_i\frac{\partial}{\partial u^+_i} - u^-_i\frac{\partial}{\partial u^-_i}\,, \tag{3.5}$$

which satisfy the SU(2) algebra

$$[D^0, D^{\pm\pm}] = \pm 2D^{\pm\pm}\,, \qquad [D^{++}, D^{--}] = D^0\,. \tag{3.6}$$

Likewise, the usual space-time and Grassmann integration measures are introduced, as well as a measure du for integration over the sphere satisfying

$$\int du = 1\,, \qquad \int du\, f^{(q)}(u) = 0 \ \text{ if } q \neq 0\,, \qquad \int du\, u^+_{(i_1}\cdots u^+_{i_n}u^-_{j_1}\cdots u^-_{j_n)} = 0\,, \tag{3.7}$$

where $f^{(q)}$ is any field of U(1) charge q.

It is useful to introduce the harmonic-projected Grassmann variables

$$\theta^\pm_\alpha \equiv u^\pm_i\theta^i_\alpha\,, \qquad \overline{\theta}^\pm_{\dot\alpha} \equiv u^\pm_i\overline{\theta}^i_{\dot\alpha}\,, \tag{3.8}$$

their derivatives,

$$\partial^\pm_\alpha \equiv \frac{\partial}{\partial\theta^{\mp\alpha}} = \pm u^\pm_i\frac{\partial}{\partial\theta^\alpha_i}\,, \qquad \overline{\partial}^\pm_{\dot\alpha} \equiv \frac{\partial}{\partial\overline{\theta}^{\mp\dot\alpha}} = \pm u^{\pm i}\frac{\partial}{\partial\overline{\theta}^{\dot\alpha i}}\,, \tag{3.9}$$

and the associated harmonic-projected covariant derivatives

$$\begin{aligned} D^\pm_\alpha &\equiv u^\pm_i D^i_\alpha = \pm\partial^\pm_\alpha + i\overline{\theta}^{\pm\dot\alpha}\,\partial\!\!\!/_{\alpha\dot\alpha}\,, \\ \overline{D}^\pm_{\dot\alpha} &\equiv -u^{\pm i}\overline{D}_{\dot\alpha i} = \pm\overline{\partial}^\pm_{\dot\alpha} - i\theta^{\pm\alpha}\,\partial\!\!\!/_{\alpha\dot\alpha}\,. \end{aligned} \tag{3.10}$$

We use the set $\{x^\mu, \theta^\pm_\alpha, \overline{\theta}^\pm_{\dot\alpha}, u^\pm_i\}$ as a coordinate basis — called the *central basis*[3] — for harmonic superspace. Notice that in changing basis from the θ^i's to the $\theta^\pm$'s the harmonic derivatives (3.5) pick up extra terms, e.g., $D^{++} = u^+_i(\partial/\partial u^-_i) + \theta^{+\alpha}\partial^+_\alpha + \overline{\theta}^{+\dot\alpha}\overline{\partial}^+_{\dot\alpha}$, etc. The harmonic covariant derivatives then obey together with (3.6) the algebra

$$\begin{aligned} [D^{\pm\pm}, D^\mp_\alpha] = D^\pm_\alpha\,, \qquad [D^{\pm\pm}, \overline{D}^\mp_{\dot\alpha}] = \overline{D}^\pm_{\dot\alpha}\,, \\ \{D^\pm_\alpha, \overline{D}^\mp_{\dot\alpha}\} = \mp 2i\,\partial\!\!\!/_{\alpha\dot\alpha}\,, \end{aligned} \tag{3.11}$$

with all other (anti)commutators vanishing. Eqs. (3.11) and (3.6) give the form of the $N=2$ algebra on harmonic superspace that we will use.

[3]Note that this is slightly different from the meaning of central basis in [24] who use the θ^i's instead of the $\theta^\pm$'s.

$N = 2$ supersymmetry invariants can be formed by integrating a general harmonic superfield over all the superspace coordinates with measure $\int du\, d^4x\, d^4\theta^+\, d^4\theta^-$, where, up to total space-time derivatives,

$$\int d^4\theta^\mp = (D^\pm)^4 \equiv \frac{1}{16}(D^\pm)^2(\overline{D}^\pm)^2\,, \tag{3.12}$$

where the derivatives are evaluated at $\theta^\mp = 0$.

Two different constraints in $N = 2$ harmonic superspace can be used to reduce superfield representations. We refer to these two conditions as the *chiral constraint* and the (Grassmann) *analytic constraint*, respectively.

The chiral constraint on a general superfield Φ,

$$\overline{D}^+_{\dot\alpha}\Phi = \overline{D}^-_{\dot\alpha}\Phi = 0\,, \tag{3.13}$$

is consistent since $\{\overline{D}^+_{\dot\alpha}, \overline{D}^-_{\dot\beta}\} = 0$, and can be solved by introducing the chiral space-time coordinate

$$x^\mu_C \equiv x^\mu - i\theta^+\sigma^\mu\overline{\theta}^- + i\theta^-\sigma^\mu\overline{\theta}^+ \tag{3.14}$$

annihilated by $\overline{D}^\pm$. Then, in the *chiral basis* $\{x^\mu_C, \theta^\pm_\alpha, \overline{\theta}^\pm_{\dot\alpha}, u^\pm_i\}$, the chiral constraint can be solved by an arbitrary (unconstrained) superfield independent of the $\overline{\theta}^\pm$'s: $\Phi = \Phi(x^\mu_C, \theta^\pm_\alpha, u^\pm_i)$. These *chiral superfields* are useful for describing the field-strength superfield for the vector multiplet. Supersymmetry invariants can be constructed by integrating chiral superfields against the measure

$$\int du\, d^4x_C\, d^4\theta = \int du\, d^4x\, D^4\,, \tag{3.15}$$

where

$$D^4 \equiv \frac{1}{16}(D^+)^2(D^-)^2\,. \tag{3.16}$$

The analytic constraint on a general superfield Φ,

$$D^+_\alpha\Phi = \overline{D}^+_{\dot\alpha}\Phi = 0\,, \tag{3.17}$$

is consistent since $\{D^+_\alpha, \overline{D}^+_{\dot\alpha}\} = 0$, and can be solved by introducing the analytic space-time coordinate

$$x^\mu_A \equiv x^\mu - i\theta^+\sigma^\mu\overline{\theta}^- - i\theta^-\sigma^\mu\overline{\theta}^+ \tag{3.18}$$

annihilated by D^+ and $\overline{D}^+$. Then, in the *analytic basis* $\{x^\mu_A, \theta^\pm_\alpha, \overline{\theta}^\pm_{\dot\alpha}, u^\pm_i\}$, the analytic constraint can be solved by an arbitrary (unconstrained) superfield independent of θ^- and $\overline{\theta}^-$: $\Phi = \Phi(x^\mu_A, \theta^+_\alpha, \overline{\theta}^+_{\dot\alpha}, u^\pm_i)$. These *analytic superfields* are useful for describing the hypermultiplet as well as the vector potential superfield for the vector multiplet. Supersymmetry invariants can be constructed by integrating analytic superfields against the measure

$$\int du\, d^4x_A\, d^4\theta^+ = \int du\, d^4x\, (D^-)^4\,, \tag{3.19}$$

where $(D^-)^4$ is defined in (3.12).

Finally, the reality conditions on superspace actions can be deduced from the action (3.1) and (3.3) of complex conjugation on the coordinates. In addition, one can introduce another kind of conjugation, called *tilde conjugation*, by combining complex conjugation with the antipodal map on the 2-sphere. Thus

$$\widetilde{\theta^i_\alpha} = -\overline{\theta}_{\dot\alpha i}\,, \qquad \widetilde{u^{\pm i}} = -u^{\pm}_i\,, \tag{3.20}$$

from which the tilde conjugation properties of $\theta^\pm$, $D^\pm$, and $D^{\pm\pm}$ can be deduced. These properties for both complex and tilde conjugation are summarized in appendix A.4 of [24]. In particular, $\widetilde{D^+} = -\overline{D}^+$ and $\widetilde{\overline{D}^+} = D^+$, so that tilde conjugation preserves analytic superfields. Notice also that $\overline{x_C} = \widetilde{x_C}$, while $x_A = \widetilde{x_A}$.

4. Higgs branch terms

Hypermultiplets are described by scalar analytic superfields of U(1) charge +1, traditionally called q^+. The bosonic terms in the component expansion of the analytic superfield q^+ and its conjugate $\tilde{q}^+$ are, in the analytic basis,

$$\begin{aligned} q^+ &= F^+ + (\theta^+)^2 M^- + (\overline{\theta}^+)^2 N^- + i\theta^+\sigma^\mu\overline{\theta}^+ A^-_\mu + (\theta^+)^2(\overline{\theta}^+)^2 P^{(-3)}\,, \\ \tilde{q}^+ &= \widetilde{F}^+ + (\theta^+)^2 \widetilde{M}^- + (\overline{\theta}^+)^2 \widetilde{N}^- + i\theta^+\sigma^\mu\overline{\theta}^+ \widetilde{A}^-_\mu + (\theta^+)^2(\overline{\theta}^+)^2 \widetilde{P}^{(-3)}\,, \end{aligned} \tag{4.1}$$

where F^+, M^-, N^-, A^-_μ, and $P^{(-3)}$ are functions of x^μ_A and the u's, and the tildes on these functions act as complex conjugation on the coefficient functions of their u-expansion, while acting as SU(2) conjugation on the u's as in (3.20). For example, the u-expansion of F^+ and $\widetilde{F}^+$ are

$$\begin{aligned} F^+ &= f^i(x_A)\, u^+_i + f^{(ijk)}(x_A)\, u^+_i u^+_j u^-_k + \cdots\,, \\ \widetilde{F}^+ &= \overline{f}_i(x_A)\, u^{+i} + \overline{f}_{(ijk)}(x_A)\, u^{+i} u^{+j} u^{-k} + \cdots\,. \end{aligned} \tag{4.2}$$

It turns out, as we will see below, that the f^i's are the propagating complex scalar fields of the hypermultiplet. As was argued in the introduction, f (and all propagating scalars, generally) should be assigned dimension 0 in the derivative expansion. Since the 2-sphere coordinates $u^\pm$ were also assigned dimension 0, we see that F^+ and therefore q^+ have dimension 0.

Since $[d\theta] = 1/2$, we see that integrating arbitrary functions of the q^+ or $\tilde{q}^+$ analytic superfields against the analytic measure (3.19) gives a 2-derivative term: $\int du\, d^4\theta^+\, B(q^+, \tilde{q}^+)$. However, since $[u^\pm] = 0$, it follows that $[D^{\pm\pm}] = 0$; also from the algebra of derivatives (3.11) it follows that if Φ is an analytic superfield, then so is $(D^{++})^n\Phi$. Therefore the general 2-derivative superspace action is

$$S^H_2 = \int du\, d^4x_A\, d^4\theta^+\, A(q^+, \tilde{q}^+; u^\pm, D^{++})\,. \tag{4.3}$$

The arbitrary number of D^{++} derivatives that can appear in S_2 is an example of the infinite redundancy of the harmonic superspace formalism, discussed in the introduction. For the case of the 2-derivative action, this redundancy has been solved in the sense that it has been shown [24] that any 2-derivative action of hypermultiplets can be realized by A's of the more specific form

$$A = -\tilde{q}^+ D^{++} q^+ + L^{+4}(q^+, \tilde{q}^+, u^\pm)\,, \tag{4.4}$$

where L^{+4} is an arbitrary local functional with no dependence on the D^{++} derivatives and of total U(1) charge +4. Note that as q^+ is complex, $\tilde{q}^+$, which contains the complex conjugate of the component fields of q^+, should be varied independently. (Alternatively, one could treat q^+ and its complex conjugate q^- as independent fields. Then the form of S_2 would be quite complicated as $\tilde{q}^+$ is given by a non-local expression involving an infinite series of $D^{\pm\pm}$ derivatives acting on q^-; see, e.g., eq. (3.111) of [24].) Also, many different hypermultiplets can easily be included by putting indices on the q's. Finally, note that the explicit u-dependence in A permits a non-SU$(2)_R$ invariant action.

Now let us examine the possible superfield form of 3- and 4-derivative terms in effective actions. Since the θ^+ integrations over the analytic subspace is of dimension 2, to get a 3-derivative term we must include derivatives acting on the hypermultiplet fields. The possibilities are either two spinor covariant derivatives or one space-time derivative. However, the supercovariant derivatives either annihilate the hypermultiplets or don't anticommute with the analytic constraints, and so do not give supersymmetry invariants upon integration over the analytic subspace. Since space-time derivatives commute with all the supercovariant derivatives, if Φ is an analytic superfield, then so is $\partial_\mu\Phi$. But a single space-time derivative cannot give rise to a Lorentz invariant term. Thus there are no 3-derivative terms on the Higgs branch.

The observation that $\partial_\mu\Phi$ is analytic if Φ is gives a simple way of making higher-derivative terms from analytic superfields by simply allowing space-time derivatives in A in (4.3). For Lorentz invariance we need an even number of space-time derivatives, so the leading term is a 4-derivative term of the form

$$S^H_{4a} = \int du\, d^4x_A\, d^4\theta^+\, \partial^\mu q^+ \partial_\mu q^+ B(q^+; u^\pm, D^{++}) + \text{c.c.}\,, \tag{4.5}$$

for an arbitrary function B, where for simplicity we have used q^+ to denote either q^+ or $\tilde{q}^+$. Only D^{++} and not D^{--} can appear because $D^{--}q^+$ is not analytic. This type of 4-derivative term seems to have been missed in other analyses of $N=2$ effective actions. A similar term, but with both space-time derivatives acting on a single field, can always be traded for a term of the form (4.5) by an integration by parts.

Another analytic combination of superfields and derivatives is $(D^+)^4 q^-$. But any action of the form $\int du\, d^4x_A d^4\theta^+\, (D^+)^4 q^-\, \widetilde{B}(q^+; u^\pm, D^{++})$ vanishes identically since a D^+ derivative annihilates $\widetilde{B}$, and so can be taken out of the whole integrand where it is annihilated by the Grassmann measure. In fact, any analytic action involving $(D^+)^4 q^-$'s, and not just the 4-derivative one written above, vanishes for the same reason.

Other higher-derivative terms can arise from lagrangians which do not obey the analytic constraint. To be supersymmetry invariants, these actions must then be integrated over all of harmonic superspace. The integrand can then contain an arbitrary function of any of the derivatives, as well as $u^\pm$, q^+ and its complex conjugate (anti-analytic) superfield

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$q^- \equiv \overline{q^+}$ (as well as their tilde-conjugates). Since the measure $du\, d^4\theta^+\, d^4\theta^-$ already has dimension four, the most general non-analytic 4-derivative term has the form

$$S_{4b}^H = \int du\, d^4x_A\, d^8\theta\, \Gamma(q^+, q^-; u^\pm, D^{\pm\pm})\,, \tag{4.6}$$

for an arbitrary function Γ, where again we use $q^\pm$ to denote also $\widetilde{q}^\pm$.

An important point in the harmonic superspace formalism reviewed above is that the auxiliary 2-sphere coordinates $u^\pm$ are not physical coordinates: they are always integrated over in any physical quantity. Therefore in writing harmonic superspace lagrangians there is no constraint of locality with respect to the $u^\pm$ coordinates. For example, terms containing both fields and tilde-conjugated fields — which involve a non-local inversion on the 2-sphere — are allowed; also 2-sphere non-locality can appear through terms having arbitrarily large numbers of $D^{\pm\pm}$ derivatives, or multiple du integrations. Indeed, for a given dimension term in an effective action expansion, like (4.5) and (4.6) above, there are infinitely many allowed harmonic superspace terms since there is no restriction on the number of $D^{\pm\pm}$'s. This non-locality is a major technical obstacle to using the harmonic superspace formalism for making systematic derivative expansions of effective actions. Nevertheless, after integrating over the 2-sphere and removing all the auxiliary fields with their equations of motion, there remain only a finite number of distinct terms of a given dimension. Thus the 2-sphere non-locality in harmonic superspace represents a redundancy in its description of supersymmetric lagrangians. In practice, this infinite redundancy means that there are infinitely many different dependencies of the action on the infinitely many auxiliary fields of harmonic superfields. But all of these actions reduce to the same action when the auxiliary fields are substituted in terms of propagating fields using their equations of motion.

Now, as was discussed in the introduction, once the form of the leading term in the derivative expansion of the effective action is fixed, then there is a systematic procedure to compute the corrected equations for the auxiliary fields order-by-order, implying that the redundancy in the form of harmonic superspace actions can be circumvented in principle, thus allowing a systematic classification and construction of higher-derivative terms in effective actions. The key point for our purposes is that given the 2-derivative term, S_2, of the superspace effective action, the 4-derivative terms are given by a dimension four superspace effective action S_4 *with auxiliary field components evaluated at their values given by* S_2, as in (1.8).

In the rest of this section we use this understanding to prove that no WZ terms can be constructed on the Higgs branch and also to show that the appearance of arbitrary combinations of $D^{\pm\pm}$'s in (4.5) and (4.6) can be brought under control.

A WZ term is a 4-derivative term where the propagating scalars ϕ^a enter in the lagrangian as

$$\lambda_{abcd}(\phi)\epsilon^{\mu\nu\rho\sigma}\partial_\mu\phi^a\partial_\nu\phi^b\partial_\rho\phi^c\partial_\sigma\phi^d\,, \tag{4.7}$$

with some (generally singular) antisymmetric coefficient function λ_{abcd}. So we wish to search for 4-derivative terms S_4 made from hypermultiplet superfields in harmonic superspace that can give rise to (4.7) in their component expansion after substituting out all the auxiliary fields. It is immediately clear that terms of the form (4.5) cannot give rise to WZ terms: when expanded in terms of propagating scalars, two of the four space-time derivatives must come from the explicit space-time derivatives in (4.5); however since they are contracted, they cannot contribute to a WZ term (4.7) where the derivatives are all antisymmetrized. Thus we search for WZ terms among terms of the form (4.6).

4.1 Free hypermultiplets

We start with free hypermultiplets to illustrate our argument in an algebraically simple setting, and later we will generalize to arbitrary hypermultiplet 2-derivative actions.

The free hypermultiplet action is [24]

$$S_2^H = -\int du\, d^4x_A\, d^4\theta^+\, \tilde{q}^+ D^{++} q^+. \tag{4.8}$$

Using the fact that in the analytic basis $D^{++} = \partial^{++} - 2i\theta^+\sigma^\mu\overline{\theta}^+\partial_\mu + \cdots$ (where we've defined $\partial^{++} \equiv u^{+i}\partial/\partial u^{-i}$), that $\int d^4\theta^+ (\theta^+)^2(\overline{\theta}^+)^2 = 1$, and using the identity $(\theta^+\sigma^\mu\overline{\theta}^+)(\theta^+\sigma^\nu\overline{\theta}^+) = \frac{1}{2}\eta^{\mu\nu}(\theta^+)^2(\overline{\theta}^+)^2$, we find the bosonic components of (4.8) are

$$\begin{aligned} S_2^H \supset -\int du\, d^4x \Big[& \widetilde{F}^+\left(\partial^{++}P^{(-3)} + \partial^\mu A_\mu^-\right) + \widetilde{M}^-\partial^{++}N^- + \widetilde{N}^-\partial^{++}M^- + \\ & + \widetilde{A}^{\mu-}\left(\partial_\mu F^+ - \frac{1}{2}\partial^{++}A_\mu^-\right) + \widetilde{P}^{(-3)}\partial^{++}F^+\Big]. \end{aligned} \tag{4.9}$$

Varying with respect to the tilded fields one finds algebraic equations of motion whose solutions are simply

$$F^+ = f^i(x_A)\, u_i^+\,, \qquad A_\mu^- = 2\partial_\mu f^i(x_A)\, u_i^-\,, \qquad M^- = N^- = P^{(-3)} = 0\,, \tag{4.10}$$

plus the free equation of motion $\partial^2 f^i = 0$ (coming from the lowest u-component of the $\widetilde{F}^+$ equation). Thus all the fields except the first component of the u-expansion of F^+ are auxiliary fields.

Since we are only interested in extracting the purely bosonic 4-derivative terms from S_{4b}^H, and given the result from the introduction that we need substitute the auxiliary fields in S_{4b}^H using their 2-derivative values, it follows that *it is sufficient to use the q^+ superfield modulo the constraints (4.10)*. It will be convenient to deal with the various complex and tilde conjugates of q^+ in parallel. Denote by q^- the complex conjugate of q^+, so the various conjugates satisfy

$$0 = D^+q^+ = \overline{D}^+q^+\,, \quad 0 = D^+\tilde{q}^+ = \overline{D}^+\tilde{q}^+\,, \quad 0 = D^-q^- = \overline{D}^-q^-\,, \quad 0 = D^-\tilde{q}^- = \overline{D}^-\tilde{q}^-\,. \tag{4.11}$$

Their bosonic component expansions modulo the free action auxiliary field equations of motion (4.10) are then, in the analytic basis,

$$\begin{aligned} q^+ &= +f^i(x_A)\, u_i^+ + 2i(\theta^+\partial\!\!\!/\overline{\theta}^+) f^i(x_A)\, u_i^-\,, \\ \tilde{q}^+ &= +\overline{f}_i(x_A)\, u^{+i} + 2i(\theta^+\partial\!\!\!/\overline{\theta}^+)\overline{f}_i(x_A)\, u^{-i}\,, \\ q^- &= -\overline{f}_i(\overline{x}_A)\, u^{-i} + 2i(\theta^-\partial\!\!\!/\overline{\theta}^-)\overline{f}_i(\overline{x}_A)\, u^{+i}\,, \\ \tilde{q}^- &= +f^i(\overline{x}_A)\, u_i^- - 2i(\theta^-\partial\!\!\!/\overline{\theta}^-) f^i(\overline{x}_A)\, u_i^+\,. \end{aligned} \tag{4.12}$$

It will be convenient to expand the expressions (4.12) in the central basis where the full θ dependence is manifest. Using (3.18) and defining the shorthands

$$\begin{aligned} \not\partial^{++} &\equiv \theta^+ \not\partial \overline{\theta}^+ \,, \qquad \not\partial^{+-} \equiv \theta^+ \not\partial \overline{\theta}^- \,, \qquad \not\partial^{-+} \equiv \theta^- \not\partial \overline{\theta}^+ \,, \qquad \not\partial^{--} \equiv \theta^- \not\partial \overline{\theta}^- \,, \\ & \qquad\qquad f^\pm \equiv f^i(x) u_i^\pm \,, \qquad \overline{f}^\pm \equiv \overline{f}_i(x) u^{\pm i} \,, \end{aligned} \tag{4.13}$$

we find

$$\begin{aligned} q^+ &= +f^+ + 2i\not\partial^{++} f^- - i\not\partial^{+-} f^+ - i\not\partial^{-+} f^+ + \mathcal{O}(\partial^2 f) \,, \\ \tilde{q}^+ &= +\overline{f}^+ + 2i\not\partial^{++} \overline{f}^- - i\not\partial^{+-} \overline{f}^+ - i\not\partial^{-+} \overline{f}^+ + \mathcal{O}(\partial^2 f) \,, \\ q^- &= -\overline{f}^- + 2i\not\partial^{--} \overline{f}^+ - i\not\partial^{+-} \overline{f}^- - i\not\partial^{-+} \overline{f}^- + \mathcal{O}(\partial^2 f) \,, \\ \tilde{q}^- &= +f^- - 2i\not\partial^{--} f^+ + i\not\partial^{+-} f^- + i\not\partial^{-+} f^- + \mathcal{O}(\partial^2 f) \,, \end{aligned} \tag{4.14}$$

where $\mathcal{O}(\partial^2 f)$ stands for terms with two or more space-time derivatives acting on f. We can neglect the 2-derivative terms acting on a single field since those derivatives are necessarily symmetrized and so can never contribute to the WZ term (4.7).

Since in the central basis $D^{\pm\pm}$ act simply as $D^{\pm\pm} u^\pm = 0$ and $D^{\pm\pm} u^\mp = u^\pm$, we get from (4.14)

$$\begin{aligned} D^{++} q^+ &= \mathcal{O}(\partial^2 f) \,, \\ D^{++} q^- &= -\overline{f}^+ - 2i\not\partial^{++} \overline{f}^- + i\not\partial^{+-} \overline{f}^+ + i\not\partial^{-+} \overline{f}^+ + \mathcal{O}(\partial^2 f) = -\tilde{q}^+ + \mathcal{O}(\partial^2 f) \,, \\ D^{--} q^+ &= +f^- - 2i\not\partial^{--} f^+ + i\not\partial^{+-} f^- + i\not\partial^{-+} f^- + \mathcal{O}(\partial^2 f) = +\tilde{q}^- + \mathcal{O}(\partial^2 f) \,, \\ D^{--} q^- &= \mathcal{O}(\partial^2 f) \,, \end{aligned} \tag{4.15}$$

where we used $[D^{++}, \not\partial^{+-}] = \not\partial^{++}$ etc. This means that as far as the WZ terms are concerned, we need only consider the four fields $q^\pm$ and $\tilde{q}^\pm$, since $D^{\pm\pm}$ acting on them gives back the same four fields up to higher space-time derivative terms when the auxiliary fields are put on shell. This is an example of the identities relating superfields evaluated at $a = a_2$ to some number of their $\partial/\partial u$ derivatives referred to in the introduction: (4.15) shows that all derivatives of hypermultiplet fields can be reduced to one of four possibilities as far as the WZ term is concerned.

Thus, with complete generality, we can take any potential hypermultiplet WZ term to be of the form

$$S_4 = \int du\, d^4x\, d^8\theta\, \Gamma(q^+, q^-, \tilde{q}^+, \tilde{q}^-; u^\pm) \,, \tag{4.16}$$

for an arbitrary real function Γ, with no $D^{\pm\pm}$ dependence. Note that $\tilde{q}^\pm$ should not be thought of as fields to be varied independently of $q^\pm$.

More generally, if we wanted to classify all the different non-WZ 4-derivative terms, we would have to include more — though still a finite number — possibly distinct combinations of u-derivatives on fields. Indeed, it is not hard to show[4] that, using the auxiliary

[4] Including the fermions and using the 2-derivative equations of motion for the auxiliary fields, we have, in the analytic basis, $q^+ = f^+ + 2i\not\partial^{++} f^- + \theta^+\psi + \overline{\theta}^+\overline{\psi}$. In the analytic basis $D^{++} = \partial^{++} + \theta^{+\alpha}\partial^+_\alpha + \overline{\theta}^{+\dot\alpha}\overline{\partial}^+_{\dot\alpha} - 2i\not\partial^{++}$. Direct computation then gives $D^{++}q^+ = 4(\not\partial^{++})^2 f^- - 2i\not\partial^{++}(\theta^+\psi + \overline{\theta}^+\overline{\psi})$, $(D^{++})^2 q^+ = 4(\not\partial^{++})^2 f^+$, and thus $(D^{++})^3 q^+ = 0$.

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component equations of motion, $(D^{++})^3 q^+ = 0$. This implies that in a central basis expansion of (4.12), keeping all the higher-derivative as well as fermionic terms, the u-expansion of q^+ contains only the three terms u^+, $(u^+)^2 u^-$, and $(u^+)^3 (u^-)^2$. So $(D^{--})^4 q^+ = 0$ as well. Thus there are only six non-vanishing combinations of $D^{\pm\pm}$ derivatives acting on q^+. For the holomorphic term, (4.5), only q^+ and D^{++} appear, so there are only three combinations to consider,

$$q^+, D^{++} q^+, (D^{++})^2 q^+ \,, \tag{4.17}$$

while for the non-holomorphic term, (4.6), where $q^\pm$ and $D^{\pm\pm}$ may appear there are twelve non-vanishing combinations

$$\begin{array}{cccccc} (D^{--})^3 q^+ \,, & (D^{--})^2 q^+ \,, & D^{--} q^+ \,, & q^+ \,, & D^{++} q^+ \,, & (D^{++})^2 q^+ \,, \\ (D^{++})^3 q^- \,, & (D^{++})^2 q^- \,, & D^{++} q^- \,, & q^- \,, & D^{--} q^- \,, & (D^{--})^2 q^- \,. \end{array} \tag{4.18}$$

Only the three combinations (4.17) need be considered in (4.5), and only the twelve combinations (4.18) need be considered in (4.6). This leads to a finite classification of 4-derivative terms on the Higgs branches.

Returning to our search for the WZ term, we expand (4.16) by inserting the expressions (4.14) and keeping four of the derivative terms. To survive the $d^8\theta$ integration and to get a $\mathrm{tr}(\sigma^\mu \overline{\sigma}^\nu \sigma^\rho \overline{\sigma}^\sigma)$ (so that we get an $\epsilon^{\mu\nu\rho\sigma}$) we need one of each type of derivative term: $\partial\!\!\!/^{++} \partial\!\!\!/^{+-} \partial\!\!\!/^{--} \partial\!\!\!/^{-+}$. From (4.14), we see that for every field the $\partial\!\!\!/^{+-}$ and $\partial\!\!\!/^{-+}$ contributions always enter together in the form

$$(\partial\!\!\!/^{+-} + \partial\!\!\!/^{-+}) A \,, \tag{4.19}$$

where A stands for some field. Therefore in the expansion of (4.16), contributions to potential WZ terms will always appear in the combination

$$\int du\, d^4x\, d^8\theta\, (\partial\!\!\!/^{+-} + \partial\!\!\!/^{-+}) A\, (\partial\!\!\!/^{+-} + \partial\!\!\!/^{-+}) B\, \partial\!\!\!/^{++} C\, \partial\!\!\!/^{--} D \,, \tag{4.20}$$

for some A, B, C, and D. Doing the θ integrals and keeping only the $\epsilon^{\mu\nu\rho\sigma}$ piece from the σ trace, we have

$$\int du\, d^4x\, \epsilon^{\mu\nu\rho\sigma} (\partial_\mu A \partial_\nu B + \partial_\nu A \partial_\mu B) \partial_\rho C \partial_\sigma D \,, \tag{4.21}$$

which vanishes by antisymmetry. Therefore all the potential WZ terms vanish for free hypermultiplets.

Actually, we did not need the identities (4.15) to reach this conclusion. It is enough to note that in q^+ the derivatives of the scalar fields enter only in the three combinations $\partial\!\!\!/^{++} f$, $\partial\!\!\!/^{--} f$, and $(\partial\!\!\!/^{+-} + \partial\!\!\!/^{-+}) f$, which form a triplet SU(2) representation. Thus these combinations close among themselves under u-differentiation,

$$\begin{aligned} D^{\pm\pm}(\partial\!\!\!/^{+-} + \partial\!\!\!/^{-+}) &= 2\, \partial\!\!\!/^{\pm\pm} \,, \\ D^{\pm\pm}(\partial\!\!\!/^{\pm\pm}) &= 0 \,, \\ D^{\pm\pm}(\partial\!\!\!/^{\mp\mp}) &= \partial\!\!\!/^{+-} + \partial\!\!\!/^{-+} \,. \end{aligned} \tag{4.22}$$

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It follows that only these combinations can occur in the expansion of the most general 4-derivative action (4.6), and so, by arguments of the previous paragraph, cannot give rise to a WZ term.

Alternatively, one can check this argument by a direct calculation to extract a WZ-type term from (4.16). Up to a total space-time derivative, the θ integrations can be replaced by supercovariant differentiation evaluated at $\theta = 0$. This differentiation gives a total derivative for the WZ-like terms:

$$
\begin{aligned}
S_4 &= \frac{1}{16}\int du\, d^4x \left[\Gamma_{\overline{I}\overline{J}KL}\, \overline{D}^{+\dot\alpha} D^{+\alpha} q_I^- \; \overline{D}^+_{\dot\alpha} D^{-\beta} q_L^+ \; \overline{D}^{-\dot\beta} D^-_\beta q_K^+ \; \overline{D}^-_{\dot\beta} D^+_\alpha q_J^-\right]_{\theta=0} \\
&= \int du\, d^4x\, \Gamma_{\overline{I}\overline{J}KL}\, \epsilon^{\mu\nu\rho\sigma}\, \partial_\mu f_L^+\, \partial_\nu f_K^-\, \partial_\rho \overline{f}_{\overline{I}}^-\, \partial_\sigma \overline{f}_{\overline{J}}^+ \\
&= \int du\, d^4x\; \partial_\mu \left(\Gamma_{\overline{I}\overline{J}K}\epsilon^{\mu\nu\rho\sigma}\, \partial_\nu f_K^-\, \partial_\rho \overline{f}_{\overline{I}}^-\, \partial_\sigma \overline{f}_{\overline{J}}^+\right),
\end{aligned} \tag{4.23}
$$

where the subscripts denote differentiation of Γ with respect to its arguments. We are using a notation, to be introduced shortly, in which subscripts I, J, K, L label both the hypermultiplets and their tilde conjugates, so that the indices run from 1 to $2n$ where n is the number of hypermultiplets.

4.2 General hyperkahler geometry

We now extend the above argument to general 2-derivative terms for hypermultiplets (4.4) following the notation of section 11.4 of [24]. Consider a theory with n massless neutral hypermultiplets q_I^+, $I = 1, \ldots, n$. Instead of treating the $\widetilde{q}_I^+$'s separately, it is convenient to double the number of fields, letting I run from 1 to $2n$, and to impose the condition

$$
\widetilde{q_I^+} \equiv q^{+I} = \Omega^{IJ} q_J^+ , \tag{4.24}
$$

where Ω^{IJ} is the antisymmetric $\mathrm{Sp}(2n)$ invariant tensor which has the matrix form $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ in $n \times n$ blocks. Then the general hypermultiplet 2-derivative action can be written

$$
S_2^H = \int du\, d^4x_A\, d^4\theta^+ \frac{1}{2}\left(q_I^+ D^{++} q^{+I} + L^{(+4)}(q^+, u^\pm)\right), \tag{4.25}
$$

where $L^{(+4)}$ is an arbitrary function of the q^+'s and the $u^\pm$'s.

Inserting the θ-expansion (4.1) of q^+ and doing the θ integration one finds the bosonic part of the action to be

$$
\begin{aligned}
S_2^H = \int du\, d^4x_A \Big[& F_I(\partial^{++}P^I + \partial^\mu A_\mu^I) - \frac{1}{4}A_I^\mu \partial^{++} A_\mu^I + M_I \partial^{++} N^I + \\
& + \frac{1}{2}P^I \partial_I L - \frac{1}{8}(A^{\mu I} A_\mu^J - 4M^I N^J)\partial_I \partial_J L \Big],
\end{aligned} \tag{4.26}
$$

where we have dropped the U(1) charge superscripts on the F^+, A_μ^-, M^-, N^-, and $P^{(-3)}$ component fields to reduce clutter, and where $L = L^{(+4)}(F^+, u^\pm)$ and $\partial_a = \partial/\partial F^{+a}$. The

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equations of motion following from this action are

$$
\begin{aligned}
\partial^{++} F_I &= \frac{1}{2}\partial_I L\,,\\
\mathcal{D}^{++I}{}_J M^J &= \mathcal{D}^{++I}{}_J N^J = 0\,,\\
\mathcal{D}^{++I}{}_J A^J_\mu &= 2\partial_\mu F^I\,,\\
\mathcal{D}^{++I}{}_J P^J &= -\partial^\mu A^I_\mu - \frac{1}{8}(\partial^I\partial_J\partial_K L)\left(A^{\mu J}A^K_\mu - 4M^J N^K\right), \qquad (4.27)
\end{aligned}
$$

where we have defined $\mathcal{D}^{++I}{}_J = \delta^I_J\partial^{++} - \frac{1}{2}\partial^I\partial_J L$. As in the free case, the leading term in the u-expansion of F^I, $f^{Ii}(x_A)u^+_i$, contains the propagating fields, while all the other components are auxiliary. They are determined by the above equations in terms of the f^{Ii}:

$$
\begin{aligned}
F^I &= f^{Ii}u^+_i + V^I(f,u)\,,\\
M^I &= N^I = 0\,,\\
A^I_\mu &= -2E^I_{Ji}\partial_\mu f^{Ji}\,,\\
P^I &= G^I_{Ji,Kj}\partial_\mu f^{Ji}\partial^\mu f^{Kj} + H^I_{Ji}\partial^\mu\partial_\mu f^{Ji}\,, \qquad (4.28)
\end{aligned}
$$

where the $V^I(f,u)$ u-expansion starts with $u^+_{(i}u^+_j u^-_{k)}$ and is determined by the first equation in (4.27). Here $E^I_{Ji}(f,u)$ is determined by the equation $\mathcal{D}^{++I}{}_J E^J_{Ki} = -\partial F^I/\partial f^{Ki}$, and $G(f,u)$ and $H(f,u)$ are determined by similar differential equations in u. Though explicit expressions for F, E, G, and H might be difficult to find for a given L, they are local functionals of the scalar fields f^{Ii}, but not of their derivatives.

By our general arguments from section 1, to find the purely bosonic 4-derivative terms coming from S^H_{4b} (4.6), it is sufficient to substitute q^+ modulo the constraints (4.28). Furthermore, we can neglect P^I since it is proportional to 2-derivative terms with Lorentz indices contracted. This can never contribute to the WZ term (4.7). Thus, the bosonic components of q^+_I and their complex conjugates q^-_I are (we are putting the $U(1)$ charge superscripts back on F^+ and E^- now)

$$
\begin{aligned}
q^{+I} &= F^{+I}(f,u) - 2iE^{-I}{}_{Ji}(f,u)\,\not\partial^{++}f^{iI} + \mathcal{O}(\partial^2 f)\,,\\
q^{-I} &= F^{-I}(\overline{f},u) + 2iE^{+Ii}{}_J(\overline{f},u)\,\not\partial^{--}\overline{f}^I_i + \mathcal{O}(\partial^2 f)\,, \qquad (4.29)
\end{aligned}
$$

where we have defined $F^- = \overline{F^+}$ and $E^+ = \overline{E^-}$. So far we have been working in the analytic basis, where f^{iI} is a function of x^μ_A. Using (3.18), we convert to the central basis where the full θ dependence is manifest:

$$
\begin{aligned}
q^{+I} &= F^{+I} - 2iE^{-I}{}_{Ji}\not\partial^{++}f^{iJ} + i\mathcal{D}^{++I}{}_J E^{-J}{}_{Ki}(\not\partial^{+-} + \not\partial^{-+})f^{iK} + \mathcal{O}(\partial^2 f)\,,\\
q^{-I} &= F^{-I} + 2iE^{+Ii}{}_J\not\partial^{--}\overline{f}^J_i - i\mathcal{D}^{--I}{}_J E^{+Ji}{}_K(\not\partial^{+-} + \not\partial^{-+})\overline{f}^K_i + \mathcal{O}(\partial^2 f)\,, \qquad (4.30)
\end{aligned}
$$

where we have used the definition of $E^{-I}{}_{Ji}$ and its complex conjugate.

Notice that $q^\pm$ depend on the derivatives of the scalars only through the SU(2) triplet combinations $\not\partial^{\pm\pm}$ and $\not\partial^{+-} + \not\partial^{-+}$ just as in the free hypermultiplet case. Then the argument of the previous subsection again shows that no WZ term can be generated with hypermultiplets alone.

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5. Coulomb branch terms

The unconstrained $N = 2$ vector multiplet superfield is a U(1)-charge $+2$ analytic superfield V^{++}, satisfying a reality condition

$$\widetilde{V^{++}} = V^{++} \tag{5.1}$$

and transforming under U(1) gauge transformations as

$$\delta V^{++} = -D^{++}\lambda\,, \tag{5.2}$$

where λ is an arbitrary real ($\widetilde{\lambda} = \lambda$) analytic superfield of U(1)-charge 0. Though both λ and V^{++} have infinite u expansions, we can use the gauge freedom (5.2) to eliminate all but a finite number of the components of V^{++} (an analog of the Wess-Zumino gauge in $N = 1$ supersymmetry):

$$\begin{aligned} V^{++} &= i\sqrt{2}\phi(x_A)(\overline{\theta}^+)^2 - i\sqrt{2}\overline{\phi}(x_A)(\theta^+)^2 - 2iA_\mu(x_A)\theta^+\sigma^\mu\overline{\theta}^+ + \\ &\quad + 4(\overline{\theta}^+)^2\theta^{+\alpha}\psi^i_\alpha(x_A)u^-_i - 4(\theta^+)^2\overline{\theta}^+_{\dot{\alpha}}\overline{\psi}^{\dot{\alpha}i}(x_A)u^-_i + 3(\theta^+)^2(\overline{\theta}^+)^2 D^{ij}(x_A)u^-_i u^-_j\,, \end{aligned} \tag{5.3}$$

where D^{ij} are real scalars, ϕ, ψ^i_α, and D^{ij} are gauge invariant, and the real vector A_μ transforms under a residual gauge invariance in the usual way as $\delta A_\mu = \partial_\mu \ell$ for ℓ an arbitrary real function.

5.1 The field strength superfield in $N = 2$ superspace

The gauge invariant field strength superfield is constructed as follows. First, another gauge potential superfield V^{--} is defined in terms of V^{++} as the solution to the differential equation in $u^\pm$

$$D^{++}V^{--} = D^{--}V^{++}\,, \tag{5.4}$$

which has a unique solution by virtue of the harmonicity requirement on the u-sphere. V^{--} is not an analytic (or anti-analytic) superfield, but is real $V^{--} = \widetilde{V^{--}}$ and transforms under gauge transformations as $\delta V^{--} = -D^{--}\lambda$. The field strength superfield is then defined by

$$W = -\frac{1}{4}(\overline{D}^+)^2 V^{--}\,. \tag{5.5}$$

It is a straight forward exercise, using the $N = 2$ algebra (3.6) and (3.11), to check that W is gauge invariant, chiral

$$\overline{D}^\pm W = 0\,, \tag{5.6}$$

satisfies the Bianchi identities

$$D^{i\alpha}D^j_\alpha W = \overline{D}^i_{\dot{\alpha}}\overline{D}^{j\dot{\alpha}}\overline{W}\,, \tag{5.7}$$

and is u-independent

$$D^{\pm\pm}W = 0\,. \tag{5.8}$$

Thus, in expressions involving the field strength superfields alone (i.e., no $V^{\pm\pm}$'s), the integration over the auxiliary u-sphere can be done separately, leaving an expression in standard $N=2$ superspace with coordinates $\{x^\mu, \theta^\pm_\alpha, \overline{\theta}^\pm_{\dot\alpha}\}$.

Thus $W = W(x_C, \theta^\pm)$, and the component expansion of W starts with the complex scalar ϕ introduced in (5.3):

$$W(x_C, \theta^\pm) = i\sqrt{2}\phi(x_C) + \cdots . \tag{5.9}$$

Thus the derivative dimensions of W and $V^{\pm\pm}$ are

$$[W] = 0, \qquad [V^{\pm\pm}] = -1\,, \tag{5.10}$$

where the second follows from (5.4) and (5.5).

If we assume that all $N=2$ supersymmetric expressions on the Coulomb branch can be written solely in terms of the field strength superfield W, its complex conjugate, and derivatives, and is local in $N=2$ superspace, then the general form of higher-derivative terms in the effective action is easy to obtain. So, in the remainder of this subsection we will make these two assumptions, and classify the terms up to four derivatives. But, in the next subsection we will revisit these assumptions and find that the interplay between gauge invariance and locality in superspace is algebraically complicated.

With these simplifying assumptions, the leading term in the derivative expansion of the low energy effective action is the 1-derivative Fayet-Iliopoulos term

$$S_1^C = \int d^4x\, d\theta^i \cdot d\theta^j \xi_{ij} W + \text{c.c.}\,, \tag{5.11}$$

where ξ_{ij} are an SU(2) triplet of real constants. Though this is an integral over only 1/4 of superspace, it is $N=2$ invariant by virtue of the extra constraint (5.7) that W satisfies. This constraint does not lead to any other local supersymmetry invariants, so higher-derivative terms can be constructed by treating W as an unconstrained chiral superfield. Then the general 2-derivative term is the well-known holomorphic pre-potential term given by an integral over the chiral half of superspace,

$$S_2^C = \int d^4x\, d^4\theta\, \mathcal{F}(W) + \text{c.c.} \tag{5.12}$$

In close analogy to our discussion of the 3- and 4-derivative terms for the hypermultiplets in the paragraphs following (4.4), but for chiral fields instead of analytic ones, we find that there are no 3-derivative terms, and two independent 4-derivative terms:

$$\begin{aligned} S_{4a}^C &= \int d^4x\, d^4\theta\; \partial_\mu W \partial^\mu W\, \mathcal{G}(W) + \text{c.c.}, \\ S_{4b}^C &= \int d^4x\, d^4\theta\, d^4\overline{\theta}\; \mathcal{H}(W, \overline{W})\,. \end{aligned} \tag{5.13}$$

Unlike the hypermultiplet case, since there is no u-dependence in these terms, there is no redundancy coming from arbitrary $D^{\pm\pm}$ derivatives. The holomorphic S_{4a}^C term seems to have been ignored in the literature.

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Note that the subset of S^C_{4a}-type terms which can be written using integration by parts as $\int d^4x d^4\theta\, \partial^2 W\, \mathcal{J}(W)$ are actually special cases of S^C_{4b} terms, by virtue of the constraints (5.7). This follows because for an S^C_{4b} term with $\mathcal{H}$ of the special form $\overline{W}\mathcal{J}(W)$ we have

$$
\begin{aligned}
4\int d^4x\, d^8\theta\, \overline{W}\, \mathcal{J}(W) &= \frac{1}{4}\int d^4x\, d^4\theta \left[(\overline{D}^+)^2(\overline{D}^-)^2 \left(\overline{W}\, \mathcal{J}(W)\right)\right]_{\overline{\theta}=0} \\
&= \frac{1}{4}\int d^4x\, d^4\theta\, \mathcal{J}(W) \left[(\overline{D}^+)^2(\overline{D}^-)^2 \overline{W}\right]_{\overline{\theta}=0} \\
&= \frac{1}{4}\int d^4x\, d^4\theta\, \mathcal{J}(W) \left[(\overline{D}^+)^2(D^-)^2 W\right]_{\overline{\theta}=0} \qquad (5.14)\\
&= 2\int d^4x\, d^4\theta\, \mathcal{J}(W)\, [\partial^2 W]_{\overline{\theta}=0} = \int d^4x\, d^4\theta\, \mathcal{J}(W)\, \partial^2 W + \text{c.c.}
\end{aligned}
$$

In the first line we replaced the antichiral integrations by supercovariant derivatives evaluated at $\overline{\theta}=0$; in the second line we used the chirality of W; in the third line we used the Bianchi identity (5.7) in the form $(\overline{D}^-)^2\overline{W} = (D^-)^2 W$; and in the last line we used the supersymmetry algebra (3.11) to commute the $\overline{D}^+$'s past the D^-'s. Examples of S^C_{4a} terms which cannot be rewritten in this way as S^C_{4b} terms require two or more vector multiplets.

In the search for $N=2$ supersymmetric WZ terms, it is clear that they will not be found in S^C_{4a} since two of the space-time derivatives are contracted, ruling out terms such as (4.7) antisymmetrized on derivatives of scalars. The remaining possibility is the integral expression over the whole superspace of the form S^C_{4b}. An expansion of W in the central basis up to first derivatives of the scalar field ϕ gives

$$
W = i\sqrt{2}\phi + \sqrt{2}(\partial\!\!\!/^{+-} - \partial\!\!\!/^{-+})\phi + \mathcal{O}(\partial^2)\,. \qquad (5.15)
$$

Indeed, the form of this expansion can be deduced without any calculation, since only $\partial\!\!\!/^{+-}$ and $\partial\!\!\!/^{-+}$, and not $\partial\!\!\!/^{++}$ and $\partial\!\!\!/^{--}$, can appear, because W has a vanishing U(1) charge, while the u-independence of W implies that only the antisymmetric combination $(\partial\!\!\!/^{+-} - \partial\!\!\!/^{-+})$ of derivatives can occur. As we mentioned in the hypermultiplet case, in order to get the $\epsilon^{\mu\nu\rho\sigma}$ tensor required for a WZ term, we need one of each type of derivative term $\partial\!\!\!/^{++}\phi$, $\partial\!\!\!/^{+-}\phi$, $\partial\!\!\!/^{--}\phi$, and $\partial\!\!\!/^{-+}\phi$. Since only one independent combination of those four derivatives appears in (5.15), we conclude that S^C_{4b} cannot contain a WZ term and thus that there is no WZ term on the Coulomb branch.

5.2 Superspace Chern-Simons-like terms and Grassmann non-locality

The above conclusions only hold modulo the two assumptions we made: (a) manifest gauge invariance and (b) Grassmann locality. Manifest gauge invariance means that all Coulomb branch terms can be written solely in terms of the field strength superfield W. Grassmann locality means that these terms are local in the $N=2$ superspace Grassmann-odd coordinates. In this subsection we will examine these assumptions, and will give simple arguments showing that either manifest gauge invariance or Grassmann locality holds, but that to show both simultaneously involves a case by case analysis at each order of the derivative expansion. It is interesting to note that this problem has nothing to do with harmonic superspace, and exists as well for $N=1$ supersymmetric theories.

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First, let's consider the issue of (a) manifest gauge invariance. The question is whether there exist *superspace Chern-Simons-like* terms, that is, terms in the effective action which are gauge invariant, but that cannot be written solely in terms of the gauge invariant field strength superfield W and its derivatives, and must also involve the potential superfield V. (The following arguments work as well for $N=1$ as $N=2$ supersymmetry, so we drop the indices on V and W; for $N=1$ supersymmetry, V is a real scalar superfield, and W_α is a chiral spinor superfield, while for $N=2$ we have seen that V^{++} is a real analytic scalar superfield, while W is a chiral scalar superfield.) Consider the general expression for a term in the effective action involving vector multiplets, schematically:

$$S^C = \int d\zeta\, f(V, D)\,, \tag{5.16}$$

where $d\zeta$ is the measure on the appropriate superspace and D denotes all the various covariant derivatives. A partial fixing of the gauge invariance (for either $N=1$ or $N=2$ vector multiplets) allows us to set all but a finite number of auxiliary fields to zero, leaving the gauge-variant vector potential, A_μ, as well as gauge invariant scalars and spinors, which we'll collectively denote by ϕ, as component fields. In this gauge we have

$$S^C = \int d^4x\, g(A_\mu, \phi, \partial_\nu)\,, \tag{5.17}$$

where g is Lorentz invariant and gauge invariant under $\delta A_\mu = \partial_\mu \ell$. Since the ϕ's are gauge invariant and there are no Chern-Simons-like terms (as opposed to *superspace* Chern-Simons-like) terms in even dimensions,[5] it follows that up to total derivatives (5.17) can be written as

$$S^C = \int d^4x\, h(F_{\mu\nu}, \phi, \partial_\rho)\,. \tag{5.18}$$

Finally, $F_{\mu\nu}$, ϕ, and their derivatives are just components of the field strength superfield W and its derivatives, so we can write

$$S^C = \int d^4x\, h\left(\int d\theta_1 j_1(W,D)\,,\, \int d\theta_2 j_2(W,D)\,, \ldots\right), \tag{5.19}$$

where the j_n are arbitrary functions of superspace covariant derivatives and W's, and the $d\theta_i$ are appropriate integration measures over the Grassmann-odd superspace coordinates. Thus we have rewritten the general vector multiplet term (5.16) solely in terms of the field strength superfield.

But (5.19) is not local in superspace. Such a superspace-local term would have just a single integral over the Grassmann-odd coordinates,

$$S^C_{\text{local}} = \int d^4x\, d\theta\ h(W, D)\,. \tag{5.20}$$

This brings us to the issue of (b) Grassmann locality. Since the Grassmann-odd θ's are not physical coordinates, there is no *a priori* reason that effective actions should be local

[5] Actually, we do not know of a proof of this "folk theorem" which states that for every gauge-invariant f there exists a g such that $\int d^{2n}x\, f(A_\mu, \partial_\nu) = \int d^{2n}x\, g(F_{\mu\nu}, \partial_\rho)$ (modulo surface terms).

in the θ's. However, for unconstrained superfields, locality in space-time together with supersymmetry invariance actually imply locality in the θ's. This argument, which is a basic reason for the usefulness of superspace, is reviewed in appendix A. Nevertheless, this observation does not allow us to conclude that the general vector multiplet term (5.19) can be written in the local form (5.20), since the field strength superfield W is constrained by the Bianchi identities and thus the locality argument does not work.

Thus, our arguments leave open the possibility that there may exist supersymmetric terms in the effective action for vector multiplets (for $N = 1$ as well as $N = 2$ supersymmetry) which can only be written in the Grassmann non-local form (5.19). We call such terms superspace Chern-Simons-like terms since, like Chern-Simons terms in odd space-time dimensions, they cannot be written in a (superspace) local form solely in terms of the field strength. Finding such superspace non-local terms is equivalent to writing the vector multiplet in component fields and checking supersymmetry invariance "by hand". Such terms are known not to exist up to but not including three derivatives in $N = 1$ and $N = 2$ theories, while it is known that the 2-derivative terms in $N = 3$ theories are in fact superspace Chern-Simons-like terms [28, 29, 30, 24]. We will report on a search for superspace Chern-Simons-like terms in $N = 1$ and $N = 2$ theories elsewhere [31]. For the remainder of this paper, though, we ignore the possibility of their existence.

6. Mixed branch terms

Terms in the effective action on the $N = 2$ mixed branch are simply terms depending on both the neutral hypermultiplets q^+ as well as the vector multiplets W. As both q^+ and W have derivative dimension 0, and each is integrated over at least half of the Grassmann-odd coordinates in $N = 2$ superspace, terms involving either superfield have minimum derivative dimension 2. However, any term involving both hyper- and vector multiplets must have dimension greater than 2 since one is chiral and the other analytic, so they cannot be integrated over the same half of superspace. Thus the minimum dimension term has three derivatives. In this section we will construct the dimension three and four terms on the mixed branches and briefly discuss some of the physics that they describe.

Any 3-derivative term must appear as an integral over 3/4 of the anticommuting coordinates, since the q^+ and W fields have derivative dimension 0. To be supersymmetric, we must choose the 3/4 of superspace to be the overlap of the chiral and analytic halves:

$$S_3^M = \int du\, d^4x\, d^2\theta^+\, d^2\overline{\theta}^+\, d^2\theta^-\;\; F(q^+, W; u^\pm, D^{++}) + \text{c.c} \tag{6.1}$$

Here F is an arbitrary U(1)-charge $+2$ function of W and q^+ (but not their complex conjugates $\overline{W}$ and q^-). Since W is $u^\pm$-independent, the D^{++} derivatives act only on the q^+'s. Just as in the discussion of the holomorphic Higgs branch term, S_{4a}^H, the D^{--} derivatives do not appear because $D^{--}q^+$ is not analytic. Also, if the 2-derivative hypermultiplet kinetic term is free, then by our previous arguments we need only consider only the combinations q^+, $D^{++}q^+$ and $(D^{++})^2 q^+$ in F.

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To see in more detail why S_3^M is supersymmetric, recall that up to total space-time derivatives the $d\theta^{\pm}$'s in the Grassmann measure can be replaced by $D^{\mp}$'s evaluated at $\theta = 0$:

$$S_3^M = \int du\, d^4x \left[(D^-)^2 (\overline{D}^-)^2 (D^+)^2\, F((D^{++})^n q^+, W, u^{\pm}) \right]_{\theta=0} + \text{c.c.} \tag{6.2}$$

This is supersymmetric if it is annihilated by all the supercovariant derivatives. Up to total space-time derivatives, it is annihilated by $D^{\pm}$ and $\overline{D}^-$ by antisymmetry (e.g. $(D^+)^3 = 0$). It is annihilated by the remaining $\overline{D}^+$ since $\overline{D}^+(D^{++})^n q^+ = 0$ by analyticity and $[\overline{D}^+, D^{++}] = 0$, $\overline{D}^+ W = 0$ by chirality, and $\overline{D}^+ u^{\pm} = 0$ by the definition (3.10) of $\overline{D}^+$.

Note that there is a second 3-derivative term given by an integral over a different three-quarters of superspace,

$$S_3^{\prime M} = \int du\, d^4x\, d^2\theta^+\, d^2\overline{\theta}^+\, d^2\overline{\theta}^- \;\; F'(q^+, \overline{W}; u^{\pm}, D^{++}) + \text{c.c.}, \tag{6.3}$$

where F' is now an arbitrary holomorphic function of q^+ and $\overline{W}$.

These leading 3-derivative supersymmetric terms on mixed branches describe a coupling between low energy photons and the hypermultiplet and vector multiplet scalars. To see this, we calculate the bosonic part of the action S_3^M in the case where F has no D^{++} dependence (for simplicity). We calculate by distributing the covariant derivatives in (6.2) and using superfield expansions such as (4.14) and (5.15). We find that the bosonic part of the action contains the following two terms

$$\begin{aligned} S_3^{M(\text{bosonic})} = \frac{1}{8} \int du\, d^4x \Big[& F^{IJa}\, D^+_{\beta} \overline{D}^-_{\alpha} q^+_I\, \overline{D}^{-\dot\alpha} D^-_{\alpha} q^+_J\, D^{+\beta} D^{-\alpha} W_a + \\ & + \, 2\, F^{IJa}\, D^{+\alpha} D^+_{\alpha} W_a\, \overline{D}^{-\dot\alpha} D^{-\beta} q^+_I\, \overline{D}^-_{\dot\alpha} D^-_{\beta} q^+_J \Big]_{\theta=0} + \text{c.c.}, \end{aligned} \tag{6.4}$$

where the superscripts on F denote derivatives with respect to its arguments: $F^I = \partial F/\partial q^+_I$ and $F^a = \partial F/\partial W_a$, where I is an index labeling different hypermultiplets and a labels different vector multiplets. The last term in (6.4) contains $(D^+)^2 W$ which is proportional to the auxiliary D^{ij} field which vanishes by the 2-derivative equations of motion. The surviving term gives

$$S_3^{M(\text{bosonic})} = -\int du\, d^4x\;\; F^{IJa}(f^+, \phi)\, \partial_{\mu} f^+_I\, \partial_{\nu} f^-_J \left(F^{\mu\nu}_a + \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} F_{a\,\rho\sigma} \right) + \text{c.c.} \tag{6.5}$$

We now move on to the 4-derivative terms on the mixed branch. To make 4-derivative terms given as integrals over 3/4 of superspace as in S_3^M, we require derivative dimension 1 combinations covariant derivatives of q^+ and W annihilated by $\overline{D}^+$. There are five[6] such combinations: $\partial_{\mu} q^+$, $\partial_{\mu} W$, $D^+ W \cdot \sigma^{\mu} \cdot \overline{D}^- q^+$, $D^+ W \cdot D^+ W$, and $\overline{D}^- q^+ \cdot \overline{D}^- q^+$. The first three are not Lorentz invariant, so can be dropped. The second two then give rise to the

[6] Two other scalar combinations, $(\overline{D}^+)^2 \overline{W}$ and $(\overline{D}^+)^2 q^-$, also give rise to supersymmetric 4-derivative terms, but they are just special cases of the non-holomorphic 4-derivative term S_{4c}^M given in (6.7).

following 4-derivative terms:

$$
\begin{aligned}
S_{4a}^M &= \int du\, d^4x\, d^2\theta^+\, d^2\overline{\theta}^+\, d^2\theta^- \; D^+W_a \cdot D^+W_b\, G^{ab}(q^+, W; u^\pm, D^{++}) + \text{c.c.} \qquad (6.6)\\
S_{4b}^M &= \int du\, d^4x\, d^2\theta^+\, d^2\overline{\theta}^+\, d^2\theta^- \, \overline{D}^-(D^{++})^n q_I^+ \cdot \overline{D}^-(D^{++})^m q_J^+ G_{nm}^{IJ}(q^+, W; u^\pm, D^{++}) + \text{c.c.}
\end{aligned}
$$

There are also versions of each of these terms integrated over a different three-quarters of superspace, as in (6.3). Finally, there is also a non-holomorphic 4-derivative term given by an integral over all of superspace:

$$
S_{4c}^M = \int du\, d^4x\, d^8\theta\; H(q^+, q^-, W, \overline{W}; u^\pm, D^{\pm\pm})\,. \tag{6.7}
$$

The expressions (6.6) and (6.7) are non-local on the auxiliary harmonic u-sphere, since an arbitrary number of $D^{\pm\pm}$ derivatives appear. But, just as was discussed in section 1 and in section 4 following eq. (4.16), we can eliminate this non-locality by consistently using the auxiliary field equations of motion following from 2-derivative terms. In the case where the 2-derivative hypermultiplet kinetic term is free, we can limit the appearance of $D^{\pm\pm}$ to the two combinations $D^{++}q^+$ and $(D^{++})^2 q^+$ in the holomorphic terms $S_{4a,b}^M$, and to the ten combinations $(D^{++})^n q^+$, $(D^{--})^m q^+$ for $n = 1, 2$ and $m = 1, 2, 3$, and their complex conjugates in S_{4c}^M.

Which of $S_{4a,b,c}^M$ can give rise to a WZ term? Following the discussion given in the previous sections, we saw that WZ terms are proportional to $\epsilon^{\mu\nu\rho\sigma} \propto \operatorname{tr}(\sigma^\mu\overline{\sigma}^\nu\sigma^\rho\overline{\sigma}^\sigma)$, and must therefore contain all four types ($\not\partial^{++}$, $\not\partial^{+-}$, $\not\partial^{--}$, and $\not\partial^{-+}$) of derivatives acting on the scalar fields. Three independent combinations of those derivatives occurred in the hypermultiplet component expansion (4.14) and the fourth occurs in the vector multiplet component expansion (5.15). Therefore $S_{4a,b,c}^M$ could each contain a WZ term.

Recall the discussion of section 4 where it was shown that for WZ terms only $q^\pm$ and $D^{\pm\pm}q^\mp \sim \tilde{q}^\pm$ can contribute. This implies that of $S_{4a,b,c}^M$ only terms of the form

$$
\begin{aligned}
S_{4a}^{M(\mathrm{WZ})} &= \int du\, d^4x\, d^2\theta^+\, d^2\overline{\theta}^+\, d^2\theta^- \; D^+W_a \cdot D^+W_b G^{ab}(q^+, W; u^\pm) + \text{c.c.},\\
S_{4b}^{M(\mathrm{WZ})} &= \int du\, d^4x\, d^2\theta^+\, d^2\overline{\theta}^+\, d^2\theta^- \, \overline{D}^- q_I^+ \cdot \overline{D}^- q_J^+ G_{00}^{IJ}(q^+, W; u^\pm) + \text{c.c.},\\
S_{4c}^{M(\mathrm{WZ})} &= \int du\, d^4x\, d^8\theta\, H(q^+, \tilde{q}^+, q^-, \tilde{q}^-, W, \overline{W}; u^\pm)\,, \qquad (6.8)
\end{aligned}
$$

need be considered.

It is not hard to see that $S_{4a}^{M(\mathrm{WZ})}$ cannot contribute a WZ term. From (5.15) the bosonic expansion of D^+W_a is proportional to at least one factor of $\overline{\theta}^+$. Thus $(\overline{D}^-)^2$ from the superspace measure must hit the $(D^+W)^2$ factor, giving $\overline{D}^-_{\dot\alpha} D^+_\alpha W \overline{D}^{-\dot\alpha} D^{+\alpha} W \propto \partial_\mu W \partial^\mu W$. Since the space-time derivatives are contracted, this cannot give rise to a WZ term.

A similar argument shows that $S_{4b}^{M(\mathrm{WZ})}$ does not contribute a WZ term either. From (4.14), we have $\overline{D}^-_{\dot\alpha} q^+ = -2i(\theta^+\not\partial)_{\dot\alpha} f^- + \mathcal{O}(\partial^2 f)$, implying that $(D^-)^2$ from the measure must hit the $(\overline{D}^- q^+)^2$ factor to absorb the θ^+'s. Since $D^-_\alpha \overline{D}^-_{\dot\alpha} q^+ = 2i\,\partial\!\!\!/_{\alpha\dot\alpha} f^- + \mathcal{O}(\partial^2 f)$, we have $D^-_\alpha \overline{D}^-_{\dot\alpha} q^+ D^{-\alpha} \overline{D}^{-\dot\alpha} q^+ \propto \partial_\mu f^- \partial^\mu f^-$, which has contracted space-time derivatives and so cannot contribute a WZ term.

The scalar component expansion of $S_{4c}^{M(\mathrm{WZ})}$ can be performed in a similar way. Since both $q^\pm$ and $\tilde q^\pm$ appear in H, we use the compact notation (introduced in section 4.2 above) where the indices I, J, K, L run over the q's as well as the $\tilde q$'s. Thus

$$q_L^+ \equiv (q_\ell^+, -\tilde q_\ell^+)\,, \qquad q_{\overline{L}}^- \equiv (q_\ell^-, -\tilde q_\ell^-)\,. \tag{6.9}$$

For the scalar components we define

$$f_L^\pm \equiv (f_\ell^\pm, -\overline{f}_\ell^\pm)\,, \qquad \text{and} \qquad \overline{f}_{\overline{L}}^\pm \equiv (-\overline{f}_\ell^\pm, -f_\ell^\pm)\,, \tag{6.10}$$

so that $q_L^+|_{\theta=0} = f_L^+$ and $q_{\overline{L}}^-|_{\theta=0} = \overline{f}_{\overline{L}}^-$ and

$$\overline{f_L^\pm} = \pm \overline{f}_{\overline{L}}^\mp\,, \tag{6.11}$$

which follows from (3.3) and (4.13).

In order to get the right Lorentz structure, the eight supercovariant derivatives from the measure must act in $\overline{D}D$ pairs on four different fields. If we choose to act with the D's first, then none of those four fields will be a $\overline{W}$ since they are annihilated by $\overline{D}$'s. Also, by analyticity, D^+ annihilates q^+ and D^- annihilates q^-. Finally, to get the epsilon tensor, we need a trace of four sigma matrices, so that the spinor indices must be contracted to give a single trace. All these constraints mean that there are only four ways of distributing the covariant derivatives, giving

$$\begin{aligned}
S_{4c}^{M(\mathrm{WZ})} &= -\frac{1}{16}\int du\, d^4x \Big[H^{\overline{I}\overline{J}KL} \overline{D}^{+\dot\alpha} D^{+\alpha} q_{\overline{I}}^- \overline{D}^-_{\dot\beta} D^+_\alpha q_{\overline{J}}^- \overline{D}^{-\dot\beta} D^{-\beta} q_K^+ \overline{D}^+_{\dot\alpha} D^-_\beta q_L^+ + \\
&\qquad + H^{\overline{I}\overline{J}Ka} \overline{D}^{+\dot\alpha} D^{+\alpha} q_{\overline{I}}^- \overline{D}^-_{\dot\beta} D^+_\alpha q_{\overline{J}}^- \overline{D}^{-\dot\beta} D^{-\beta} q_K^+ \overline{D}^+_{\dot\alpha} D^-_\beta W_a + \\
&\qquad + H^{\overline{I}aKL} \overline{D}^{+\dot\alpha} D^{+\alpha} q_{\overline{I}}^- \overline{D}^-_{\dot\beta} D^+_\alpha W_a \overline{D}^{-\dot\beta} D^{-\beta} q_K^+ \overline{D}^+_{\dot\alpha} D^-_\beta q_L^+ + \\
&\qquad + H^{\overline{I}aKb} \overline{D}^{+\dot\alpha} D^{+\alpha} q_{\overline{I}}^- \overline{D}^-_{\dot\beta} D^+_\alpha W_a \overline{D}^{-\dot\beta} D^{-\beta} q_K^+ \overline{D}^+_{\dot\alpha} D^-_\beta W_b \Big] \\
&= 2\epsilon^{\mu\nu\rho\sigma} \int du\, d^4x \Big[i H^{\overline{I}\overline{J}KL} \partial_\mu \overline{f}_{\overline{I}}^+ \partial_\nu \overline{f}_{\overline{J}}^- \partial_\rho f_K^- \partial_\sigma f_L^+ - \\
&\qquad - \sqrt{2}\, H^{\overline{I}\overline{J}Ka}\, \partial_\mu \overline{f}_{\overline{I}}^+ \partial_\nu \overline{f}_{\overline{J}}^- \partial_\rho f_K^- \partial_\sigma \phi_a - \\
&\qquad - \sqrt{2}\, H^{\overline{I}aKL}\, \partial_\mu \overline{f}_{\overline{I}}^+ \partial_\nu \phi_a \partial_\rho f_K^- \partial_\sigma f_L^+ - \\
&\qquad - 2i\; H^{\overline{I}aKb} \partial_\mu \overline{f}_{\overline{I}}^+ \partial_\nu \phi_a \partial_\rho f_K^- \partial_\sigma \phi_b \Big]\,,
\end{aligned} \tag{6.12}$$

where the superscripts on H denote derivatives with respect to its arguments, as before. In the second equality we used that

$$\begin{aligned}
\overline{D}^\pm_{\dot\alpha} D^-_\alpha q_L^+ &= +2i\,\partial\!\!\!/_{\alpha\dot\alpha} f_L^\pm + \mathcal{O}(\partial^2 f)\,, &\qquad \overline{D}^\pm_{\dot\alpha} D^+_\alpha q_{\overline{L}}^- &= -2i\,\partial\!\!\!/_{\alpha\dot\alpha} \overline{f}_{\overline{L}}^\pm + \mathcal{O}(\partial^2 f)\,, \\
\overline{D}^\pm_{\dot\alpha} D^\mp_\alpha W_a &= \mp 2\sqrt{2}\,\partial\!\!\!/_{\alpha\dot\alpha} \phi_a + \mathcal{O}(\partial^2 \phi)\,, &\qquad \overline{D}^\pm_{\dot\alpha} D^\pm_\alpha W_a &= \mathcal{O}(\partial^2 \phi)\,,
\end{aligned} \tag{6.13}$$

which follow from the scalar field expansions (4.14) and (5.15) of $q^{\pm}$ and W to first order in derivatives. We also used the sigma matrix identity $\mathrm{tr}(\sigma^{\mu}\overline{\sigma}^{\nu}\sigma^{\rho}\overline{\sigma}^{\sigma}) = -2i\epsilon^{\mu\nu\rho\sigma} + 2\eta^{\mu\nu}\eta^{\rho\sigma} - 2\eta^{\mu\rho}\eta^{\nu\sigma} + 2\eta^{\mu\sigma}\eta^{\nu\rho}$, and kept only the $\epsilon^{\mu\nu\rho\sigma}$ piece.

The expression (6.12) can be further simplified. The fourth term in the second equality cancels by the antisymmetry on ν, σ and the symmetry on a, b. Furthermore, by (6.11) and the reality of H, it follows that the first term in the second equality in (6.12) is purely imaginary. Since the original action was real, this imaginary term must be part of a total derivative introduced when we replaced the $d\theta$ integrations by covariant derivatives. Indeed, it is not too hard to see that the first term plus the imaginary part of the second and third terms are a total derivative, and can therefore be dropped. Thus the Wess Zumino term is the real parts of the second and third terms in (6.12), which can be rewritten

$$\begin{aligned} S_{4c}^{M(\mathrm{WZ})} &= -\sqrt{2}\,\epsilon^{\mu\nu\rho\sigma}\int du\, d^4x\ \partial_\mu \overline{f}_{\overline{I}}^{+}\, \partial_\rho f_K^{-}\left(\partial_\nu \overline{f}_{\overline{J}}^{-}\,\partial^{\overline{J}} + \partial_\nu f_J^{+}\,\partial^J\right)\left(\partial_\sigma \phi_a\,\partial^a + \partial_\sigma \overline{\phi}_{\overline{a}}\,\partial^{\overline{a}}\right) H^{\overline{I}K} \\ &= 2\sqrt{2}\,\epsilon^{\mu\nu\rho\sigma}\int du\, d^4x\ \partial_\mu \overline{f}_{\overline{I}}^{+}\,\partial_\nu \overline{f}_{\overline{J}}^{-}\,\partial_\rho f_K^{-}\,\partial_\sigma f_L^{+}\, H^{\overline{I}\overline{J}KL}\,, \end{aligned} \tag{6.14}$$

where in the second line we integrated ∂_σ by parts. Since this is not a total derivative (as long as H depends on W and $\overline{W}$), we have shown that S_{4c}^{M} is the $N = 2$ supersymmetric completion of the Wess-Zumino term.

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A. Grassmann locality from space-time locality and supersymmetry

In this appendix we will present an argument showing that the combination of space-time locality with supersymmetry implies that expressions involving unconstrained superfields are necessarily local in the Grassmann-odd coordinates. Though this seems like a fundamental property of superspaces, we do not know of a reference for this argument.

To keep the notation simple, we give the argument in $N = 2$ supersymmetric quantum mechanics; the generalization to any superspace is straight forward. The superspace then consists of a space-time coordinate x and two Grassmann-odd coordinates θ and $\overline{\theta}$. Denote derivatives by

$$d = \frac{\partial}{\partial x}\,, \qquad \partial = \frac{\partial}{\partial \theta}\,, \qquad \text{and} \qquad \overline{\partial} = \frac{\partial}{\partial \overline{\theta}}\,. \tag{A.1}$$

The supercharges

$$Q = \partial - \overline{\theta} d\,, \qquad \text{and} \qquad \overline{Q} = \overline{\partial} - \theta d\,, \tag{A.2}$$

generate the supersymmetry algebra

$$\{Q, \overline{Q}\} = -2d\,, \tag{A.3}$$

and the supercovariant derivatives

$$D = \partial + \overline{\theta} d\,, \qquad \text{and} \qquad \overline{D} = \overline{\partial} + \theta d\,, \tag{A.4}$$

anticommute with the supercharges. The supercharges generate translations and super-translations of general superfields $\phi(x, \theta, \overline{\theta})$ according to

$$\delta\phi = (\alpha d + \epsilon Q + \overline{\epsilon}\overline{Q})\phi\,, \tag{A.5}$$

where α, ϵ, and $\overline{\epsilon}$ are arbitrary constants.

The general (non-local) term in an action for unconstrained superfields can be written

$$S = \int dx_1 d\theta_1 d\overline{\theta}_1 \cdots dx_n d\theta_n d\overline{\theta}_n \mathcal{K}(x_1, \theta_1, \overline{\theta}_1; \ldots ; x_n, \theta_n, \overline{\theta}_n)\phi_1(x_1, \theta_1, \overline{\theta}_1) \cdots \phi_n(x_n, \theta_n, \overline{\theta}_n)\,, \tag{A.6}$$

where $\mathcal{K}$ is an arbitrary kernel, and the ϕ_i's stand for arbitrary unconstrained superfields and their derivatives. The general action will be the sum of many such terms. If the ϕ_i's are unconstrained superfields, then super-Poincaré invariance implies $\delta S = 0$ for each term individually, since each term has a different functional dependence on the superfields. If, however, the superfields were constrained, so that there were functional relations among them, then we could only demand super-Poincaré invariance of the whole sum, not necessarily for each individual term, and the following argument would not work.

So, for unconstrained superfields we have

$$\begin{aligned} 0 = \delta S &= \int dx_1 \cdots d\overline{\theta}_n \mathcal{K}(x_1, \ldots, \overline{\theta}_n)\delta(\phi_1 \cdots \phi_n) \\ &= \int dx_1 \cdots d\overline{\theta}_n \mathcal{K}(x_1, \ldots, \overline{\theta}_n) \sum_{i=1}^{n} (\alpha d_i + \epsilon Q_i + \overline{\epsilon}\overline{Q}_i)(\phi_1 \cdots \phi_n) \\ &= -\int dx_1 \cdots d\overline{\theta}_n (\phi_1 \cdots \phi_n) \sum_{i=1}^{n} (\alpha d_i + \epsilon Q_i + \overline{\epsilon}\overline{Q}_i)\mathcal{K}(x_1, \ldots, \overline{\theta}_n)\,, \end{aligned} \tag{A.7}$$

where Q_i refers to the derivative operator (A.2) acting on $\{x_i, \theta_i \overline{\theta}_i\}$, and where in the last line we performed and integration by parts. Because α, ϵ, and $\overline{\epsilon}$ are independent arbitrary constants, and because the ϕ_i's are unconstrained, (A.7) implies that the kernel must be separately annihilated by the sums of the super-Poincaré generators:

$$0 = \left(\sum_{i=1}^{n} d_i\right)\mathcal{K} = \left(\sum_{i=1}^{n} Q_i\right)\mathcal{K} = \left(\sum_{i=1}^{n} \overline{Q}_i\right)\mathcal{K}\,. \tag{A.8}$$

The general solution to these supersymmetry equations is that $\mathcal{K}$ depends only on the combinations

$$\mathcal{K} = \mathcal{K}\left(x_i - x_j + \theta_j\overline{\theta}_i - \theta_i\overline{\theta}_j\,,\, \theta_i - \theta_j\,,\, \overline{\theta}_i - \overline{\theta}_j\right)\,. \tag{A.9}$$

Space-time locality implies that $\mathcal{K}$ must have support only for $x_i = x_j$, i.e.,

$$\mathcal{K} = \widetilde{\mathcal{K}}\left(\theta_i - \theta_j\,,\, \overline{\theta}_i - \overline{\theta}_j\right) \prod_{i>j} \delta\left(x_i - x_j + \theta_j\overline{\theta}_i - \theta_i\overline{\theta}_j\right)\,. \tag{A.10}$$

Now, by the anticommuting nature of the θ's,

$$\theta_i - \theta_j = \delta(\theta_i - \theta_j)\,, \tag{A.11}$$

and similarly for the $\overline{\theta}$'s. So any non-trivial $\widetilde{\mathcal{K}}$ factor in (A.10) just enforces (some) locality in the Grassmann-odd coordinates. Since we are trying to show just such locality, we need only concentrate on the delta-function factor in (A.10).

It is sufficient to focus on any pair of (i,j). Denote these two superspace points by $(x,\theta,\overline{\theta})$ and $(x',\theta',\overline{\theta}')$, and rename the two superfields ϕ_i, ϕ_j to ϕ and ψ, respectively. Thus we are interested in the expression

$$I = \int dx d\theta d\overline{\theta} dx' d\theta' d\overline{\theta}'\, \delta(x - x' + \theta'\overline{\theta} - \theta\overline{\theta}')\, \phi(x,\theta,\overline{\theta})\, \psi(x',\theta',\overline{\theta}') \tag{A.12}$$

for general superfields ϕ and ψ. We will show that $I = J$ where J is

$$J = \frac{1}{2}\int dx d\theta d\overline{\theta}\, \phi(x,\theta,\overline{\theta})\, [D,\overline{D}]\, \psi(x,\theta,\overline{\theta})\,, \tag{A.13}$$

and is thus local in the Grassmann-odd coordinates. Because $\phi[D,\overline{D}]\psi$ is itself another superfield, this argument can then be repeated with other pairs of superspace coordinates until the whole expression (A.6) is written as a single integral over superspace.

To show this is a straight forward computation:

$$\begin{aligned}
I &= \int dx d\theta d\overline{\theta} dx' d\theta' d\overline{\theta}'\, \phi(x,\theta,\overline{\theta})\, \delta(x - x' + \theta'\overline{\theta} - \theta\overline{\theta}')\, \psi(x',\theta',\overline{\theta}') \\
&= \int dx d\theta d\overline{\theta} dx' d\theta' d\overline{\theta}'\, \phi(x,\theta,\overline{\theta})\delta(x-x')\left\{1 + (\theta'\overline{\theta} - \theta\overline{\theta}')d' + \theta\overline{\theta}\theta'\overline{\theta}'(d')^2\right\}\, \psi(x',\theta',\overline{\theta}') \\
&= \int dx \partial\overline{\partial}\partial'\overline{\partial}' \left[\phi(x,\theta,\overline{\theta})\left\{1 + (\theta'\overline{\theta} - \theta\overline{\theta}')d + \theta\overline{\theta}\theta'\overline{\theta}' d^2\right\}\, \psi(x,\theta',\overline{\theta}')\right]_{\theta=\overline{\theta}=\theta'=\overline{\theta}'=0} \\
&= \int dx \left[\partial\overline{\partial}\phi\, \partial'\overline{\partial}'\psi + \partial\phi\, \overline{\partial}' d\psi + \overline{\partial}\phi\, \partial' d\psi + \phi d^2\psi\right]_{\theta=\overline{\theta}=\theta'=\overline{\theta}'=0} \\
&= \int dx \left[\partial\overline{\partial}\phi\, \partial\overline{\partial}\psi + \partial\phi\, \overline{\partial} d\psi + \overline{\partial}\phi\, \partial d\psi + \phi d^2\psi\right]_{\theta=\overline{\theta}=0}\,,
\end{aligned} \tag{A.14}$$

where in the second line we Taylor expanded the delta function and integrated by parts in x'; in the third line we performed the x' integration and replaced the θ integrations by derivatives evaluated at zero; in the fourth line we expanded the derivatives keeping only terms that survive when the θ's are set to zero; and in the last line we replaced θ' and $\overline{\theta}'$ with θ and $\overline{\theta}$ since they are all set to zero anyway. On the other hand,

$$\begin{aligned}
J &= \int dx d\theta d\overline{\theta}\, \phi(x,\theta,\overline{\theta})\, \frac{1}{2}[D,\overline{D}]\, \psi(x,\theta,\overline{\theta}) \\
&= \int dx d\theta d\overline{\theta}\, \phi\, (\partial\overline{\partial} - \theta\partial d + \overline{\theta}\overline{\partial} d - \theta\overline{\theta} d^2)\, \psi \\
&= \int dx\, \partial\overline{\partial}\left[\phi\, \partial\overline{\partial}\psi - \phi\, \theta\partial d\psi + \phi\, \overline{\theta}\overline{\partial} d\psi - \phi\, \theta\overline{\theta} d^2\psi\right]_{\theta=\overline{\theta}=0} \\
&= \int dx \left[\partial\overline{\partial}\phi\, \partial\overline{\partial}\psi + \overline{\partial}\phi\, \partial d\psi + \partial\phi\, \overline{\partial} d\psi + \phi\, d^2\psi\right]_{\theta=\overline{\theta}=0}\,,
\end{aligned} \tag{A.15}$$

where in the second line we expanded $[D, \overline{D}]$ using (A.4); in the third line we replaced the θ integrations by derivatives evaluated at zero; and in the last line we expanded the derivatives keeping only terms that survive when the θ's are set to zero. Thus $I = J$.

If some of the superfields are unconstrained functions over only a subspace of the full superspace, the same type of argument applies. For example, say $\psi(x', \theta', \overline{\theta}')$ is chiral, so that $\overline{D}'\psi = 0$. Then $\psi = \psi(x'_C, \theta')$ where $x'_C = x' + \theta'\overline{\theta}'$. A typical term in the action will have the general form

$$S = \int dx d\theta d\overline{\theta}\, dx'_C d\theta' \mathcal{K}(x, \theta, \overline{\theta}; x'_C, \theta')\phi(x, \theta, \overline{\theta})\, \psi(x'_C, \theta')\,, \tag{A.16}$$

and supersymmetry will imply the same constraints (A.8) on $\mathcal{K}$ as before. The solution is different, though, since $\mathcal{K}$ only depends on $\overline{\theta}'$ through x'_C, giving that $\mathcal{K}$ is a function only of the combinations $x - x'_C + 2\theta'\overline{\theta} - \theta\overline{\theta}$ and $\theta = \theta'$. Space-time locality then implies that we are interested in the expression

$$\begin{aligned}
I &\equiv \int dx d\theta d\overline{\theta}\, dx'_C d\theta' \phi(x, \theta, \overline{\theta})\delta(x - x'_C + 2\theta'\overline{\theta} - \theta\overline{\theta})\psi(x'_C, \theta') \\
&= \int dx d\theta d\overline{\theta}\, dx'_C d\theta' \phi(x, \theta, \overline{\theta})\delta(x - x'_C)\left\{1 + (2\theta'\overline{\theta} - \theta\overline{\theta})d'_C\right\}\psi(x'_C, \theta') \\
&= \int dx \partial\overline{\partial}\partial' \left[\phi(x, \theta, \overline{\theta})\left\{1 + (2\theta'\overline{\theta} - \theta\overline{\theta})d\right\}\psi(x, \theta')\right]_{\theta=\overline{\theta}=\theta'=0} \\
&= \int dx \left[\partial\overline{\partial}\phi\, \partial'\psi + 2\partial\phi\, d\psi + \phi\, \partial' d\psi\right]_{\theta=\overline{\theta}=\theta'=0} \\
&= \int dx \left[\partial\overline{\partial}\phi\, \partial\psi + 2\partial\phi\, d\psi + \phi\, \partial d\psi\right]_{\theta=\overline{\theta}=0},
\end{aligned} \tag{A.17}$$

where we have followed the same steps as in (A.14). On the other hand,

$$\begin{aligned}
J &\equiv \int dx\theta d\overline{\theta}\phi(x, \theta, \overline{\theta}) D\, \psi(x_C, \theta) \\
&= \int dx\theta d\overline{\theta}\phi(\partial + \overline{\theta}d)\psi(x + \theta\overline{\theta}, \theta) \\
&= \int dx\partial\overline{\partial}\left[\phi\partial\psi + 2\phi\overline{\theta}d\psi\right]_{\theta=\overline{\theta}=0} \\
&= \int dx \left[\partial\overline{\partial}\phi\, \partial\psi - \phi\, \partial d\psi + 2\partial\phi\, d\psi + 2\phi\, \partial d\psi\right]_{\theta=\overline{\theta}=0},
\end{aligned} \tag{A.18}$$

where we followed the same steps as in (A.15), though it should be pointed out that inside the square brackets ∂ and d refer to derivatives of ψ with respect to its arguments — i.e. partial derivatives and not total derivatives. Thus $I = J$ and we have shown that general terms involving both chiral and non-chiral unconstrained superfields are given by local superspace expressions. Similar arguments take care of the other cases (chiral-chiral and chiral-antichiral).

References

[1] M. Dine and N. Seiberg, *Comments on higher derivative operators in some SUSY field theories*, *Phys. Lett.* **B 409** (1997) 239 [hep-th/9705057].

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[2] M. Dine and J. Gray, *Non-renormalization theorems for operators with arbitrary numbers of derivatives in $N = 4$ Yang-Mills theory*, *Phys. Lett.* **B 481** (2000) 427 [hep-th/9909020].

[3] A.A. Tseytlin and K. Zarembo, *Magnetic interactions of D-branes and Wess-Zumino terms in super Yang-Mills effective actions*, *Phys. Lett.* **B 474** (2000) 95 [hep-th/9911246].

[4] K.A. Intriligator, *Anomaly matching and a Hopf-Wess-Zumino term in 6d, $N = (2, 0)$ field theories*, *Nucl. Phys.* **B 581** (2000) 257 [hep-th/0001205].

[5] D. Nemeschansky and R. Rohm, *Anomaly constraints on supersymmetric effective lagrangians*, *Nucl. Phys.* **B 249** (1985) 157.

[6] M. Henningson, *Extended superspace, higher derivatives and* $\mathrm{SL}(2, \mathbb{Z})$ *duality*, *Nucl. Phys.* **B 458** (1996) 445 [hep-th/9507135].

[7] B. de Wit, M.T. Grisaru and M. Roček, *Nonholomorphic corrections to the one-loop $N = 2$ super Yang-Mills action*, *Phys. Lett.* **B 374** (1996) 297 [hep-th/9601115].

[8] U. Lindström, F. Gonzalez-Rey, M. Roček and R. von Unge, *On $N = 2$ low energy effective actions*, *Phys. Lett.* **B 388** (1996) 581 [hep-th/9607089].

[9] G. Chalmers, M. Roček and R. von Unge, *Monopoles in quantum corrected $N = 2$ super Yang-Mills theory*, hep-th/9612195.

[10] F. Gonzalez-Rey and M. Roček, *Nonholomorphic $N = 2$ terms in $N = 4$ SYM: 1-loop calculation in $N = 2$ superspace*, *Phys. Lett.* **B 434** (1998) 303 [hep-th/9804010].

[11] I.L. Buchbinder and S.M. Kuzenko, *Comments on the background field method in harmonic superspace: non-holomorphic corrections in $N = 4$ SYM*, *Mod. Phys. Lett.* **A 13** (1998) 1623 [hep-th/9804168].

[12] F. Gonzalez-Rey, B. Kulik, I.Y. Park and M. Roček, *Self-dual effective action of $N = 4$ super–Yang-Mills*, *Nucl. Phys.* **B 544** (1999) 218 [hep-th/9810152].

[13] E.I. Buchbinder, I.L. Buchbinder and S.M. Kuzenko, *Non-holomorphic effective potential in $N = 4$* $\mathrm{SU}(N)$ *SYM*, *Phys. Lett.* **B 446** (1999) 216 [hep-th/9810239].

[14] D.A. Lowe and R. von Unge, *Constraints on higher derivative operators in maximally supersymmetric gauge theory*, *J. High Energy Phys.* **11** (1998) 014 [hep-th/9811017].

[15] I.L. Buchbinder, S.M. Kuzenko and A.A. Tseytlin, *On low-energy effective actions in $N = 2, 4$ superconformal theories in four dimensions*, *Phys. Rev.* **D 62** (2000) 045001 [hep-th/9911221].

[16] A.T. Banin, I.L. Buchbinder and N.G. Pletnev, *Low-energy effective action of $N = 2$ gauge multiplet induced by hypermultiplet matter*, *Nucl. Phys.* **B 598** (2001) 371 [hep-th/0008167].

[17] S.M. Kuzenko and I.N. McArthur, *Effective action of $N = 4$ super Yang-Mills: $N = 2$ superspace approach*, *Phys. Lett.* **B 506** (2001) 140 [hep-th/0101127].

[18] S.M. Kuzenko and I.N. McArthur, *Hypermultiplet effective action: $N = 2$ superspace approach*, *Phys. Lett.* **B 513** (2001) 213 [hep-th/0105121].

[19] I.L. Buchbinder and E.A. Ivanov, *Complete $N = 4$ structure of low-energy effective action in $N = 4$ super Yang-Mills theories*, *Phys. Lett.* **B 524** (2002) 208 [hep-th/0111062].

[20] A.T. Banin, I.L. Buchbinder and N.G. Pletnev, *On low-energy effective action in $N = 2$ super Yang-Mills theories on non-abelian background*, *Phys. Rev.* **D 66** (2002) 045021 [hep-th/0205034].

[21] I.L. Buchbinder, E.A. Ivanov and A.Y. Petrov, *Complete low-energy effective action in $N = 4$ SYM: a direct $N = 2$ supergraph calculation*, *Nucl. Phys.* **B 653** (2003) 64 [hep-th/0210241].

[22] A.T. Banin, I.L. Buchbinder and N.G. Pletnev, *One-loop effective action for $N = 4$ SYM theory in the hypermultiplet sector: leading low-energy approximation and beyond*, hep-th/0304046.

[23] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, *Unconstrained $N = 2$ matter, Yang-Mills and supergravity theories in harmonic superspace*, *Class. and Quant. Grav.* **1** (1984) 469.

[24] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky, and E.S. Sokatchev, *Harmonic superspace*, Cambridge University Press, Cambridge 2001.

[25] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [hep-th/9407087].

[26] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099].

[27] P.C. Argyres, M. Ronen Plesser and N. Seiberg, *The moduli space of $N = 2$ SUSY QCD and duality in $N = 1$ SUSY QCD*, *Nucl. Phys.* **B 471** (1996) 159 [hep-th/9603042].

[28] A.A. Rosly, *Constraints in supersymmetric Yang-Mills theory as integrability conditions*, in *Group theory methods in physics*, M. Markov and V. Man'ko eds., Nauka, 1983, p. 263.

[29] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, *Unconstrained off-shell $N = 3$ supersymmetric Yang-Mills theory*, *Class. and Quant. Grav.* **2** (1985) 155.

[30] A.S. Galperin, E.A. Ivanov and V.I. Ogievetsky, *Superspaces for $N = 3$ supersymmetry*, *Sov. J. Nucl. Phys.* **46** (1987) 543.

[31] P.C. Argyres, A.M. Awad, G.A. Braun and F.P. Esposito, to appear.